LAVER-SHELAH ITERATION

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ABSTRACT. My personal notes from Laver and Shelah [3]: assuming a weakly compact cardinal, it is consistent that all $\aleph_2$-Aronszajn trees are special and CH holds.

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The goal is to produce a model of set theory where there are $\aleph_2$-Aronszajn trees and all of them are special. An $\aleph_2$-tree T is **special** if there is a function $f : T \to \omega_1$ such that

$$f(s) = f(t) \implies s \perp t$$

for all distinct nodes $s, t \in T$. We start by assuming GCH and the existence of a weakly compact cardinal κ and force over this model. The poset is an iteration $(\mathbb{P}_\delta : \delta \leqslant \kappa^+)$ such that the first poset collapses κ to $\aleph_2$ and the rest specialize all κ-Aronszajn trees. The existence of $\aleph_2$-Aronszajn trees in the final model is guaranteed by preserving CH along the iteration.

Indeed, the first poset $\mathbb{P}_0$ will be the collapse $\mathsf{Coll}(\omega_1, < \kappa)$, which consists of countable partial functions $p : \omega_1 \times \kappa \to \kappa$ such that $p(\alpha, \beta) \in \beta$ for all $\alpha \in \omega_1$ and $\beta \in \kappa$. This poset creates κ-Aronszajn trees, since κ becomes $\aleph_2$ and CH holds in $V^{\mathbb{P}_0}$. Each iterand $\dot{\mathbb{Q}}_\delta$ will be a $\mathbb{P}_\delta$-name for the poset of countable approximations of a specialising function for a tree given by a bookkeeping function. The final poset $\mathbb{P}_{\kappa^+}$ will be σ-closed and have κ-cc. The σ-closure guarantees preservation of CH and κ-cc guarantees preservation of κ.

Furthermore, each poset $\mathbb{P}_\delta$ for $\delta < \kappa^+$ will be **κ-strongly proper**: strongly proper with respect to stationarily many models of size $< \kappa$. By κ-strong properness, lots of new subsets of ω_1 are added, so the final model will satisfy $2^{\omega_1} = \omega_3$.

Note that it is *not possible* to specialize a κ-Aronszajn tree T which is in V with κ-strongly proper poset. Yet, the posets in this proof are κ-strongly proper. This is not a problem, since there are no κ-Aronszajn trees in V, since κ is weakly compact. All the trees appear along the iteration. This implies, in particular, that the quotient posets $\mathbb{P}_\delta/\mathbb{P}_\gamma$ are not κ-strongly proper for any $\gamma < \delta < \kappa^+$, as they indeed do specialize κ-Aronszajn trees.

The fact that there are no κ-Aronszajn trees in V allows to perform a Mitchell-style splitting argument, which is described in detail in Lemma 4.8. The argument here generalizes to wide $\aleph_2$-Aronszajn trees, with a version of Lemma 4.8 that allows to split nodes that are "exit nodes from the side", i.e. exit nodes from V_α of height $< \alpha$.

1. Strong properness

Definition 1.1. Let $\mathbb{P}$ be a poset and let M be a set.

(1) Let $p \in \mathbb{P}$. A condition $r \in \mathbb{P} \cap M$ is a **residue** for p into M if
$$\forall w \in \mathbb{P} \cap M (w \leqslant r \to w \| p).$$
(2) A condition p is **strongly** $(\mathbb{P}, M)$**-generic** if every $q \leqslant p$ has a residue into M.
(3) The poset $\mathbb{P}$ is **strongly proper with respect to** M if for every $p \in \mathbb{P} \cap M$ there is $q \leqslant p$ which is strongly $(\mathbb{P}, M)$-generic.
(4) The poset $\mathbb{P}$ is κ**-strongly proper** if it is strongly proper with respect to stationarily many $M \in \mathscr{P}_\kappa(H_\theta)$ for any large enough regular θ.

A proof of the following lemma can be found in [1]:

Lemma 1.2. *A condition p is strongly $(\mathbb{P}, M)$-generic if and only if*
$$p \Vdash \check{G} \cap M \text{ is a } V\text{-generic filter on } \mathbb{P} \cap M.$$

Thus, strong properness is a generalisation of properness.

We define the notion of *common* or *dual residue* because we will do a following kind of *splitting argument*; we are in a situation where we have a poset $\mathbb{P}$, a name for a tree $\dot{T}$, a suitable model M, a condition $p \in \mathbb{P}$ and a node $t \notin M$. We will split the node t in the following sense: we find two distinct nodes s^L and s^R in M and two conditions q^L and q^R extending p such that they have a common residue r into M and such that $q^L \Vdash s^L < t$ and $q^R \Vdash s^R < t$.

Definition 1.3. Let $\mathbb{P}$ be a poset and let M be a suitable model. Let $p, q \in \mathbb{P}$. A **common residue** for p and q in M is a condition $r \in \mathbb{P} \cap M$ which is a residue for p and residue for q, i.e.
$$\forall w \in \mathbb{P} \cap M (w \leqslant r \to w \| p \wedge w \| q).$$

Remark 1.4. If two conditions p and q have a common residue it does not follow that p and q are compatible.

2. The poset

Throughout, let κ be a fixed weakly compact cardinal. We will define a countable support iteration $(\mathbb{P}_\delta : \delta \leqslant \kappa^+)$ that will collapse κ onto $\aleph_2$ and specialize all κ-Aronszajn trees.

Notation 2.1. We fix a bookkeeping function
$$\kappa^+ \to H_{\kappa^+}, \quad \delta \mapsto \dot{T}_\delta$$
such that $\dot{T}_\delta$ is a $\mathbb{P}_\delta$-name for a κ-Aronszajn tree with domain κ, whenever the poset $\mathbb{P}_\delta$ is defined, for every $\delta < \kappa^+$.

For a tree T, a **partial specializing function** is a partial function f on T such that for any two distinct nodes $s, t \in T$, if $f(s) = f(t)$, then s and t are incompatible in the tree order of T.

Definition 2.2. For each $\delta \leqslant \kappa^+$, conditions of the poset $\mathbb{P}_\delta$ are functions

$$p : \delta \to V_\kappa$$

such that

(1) $p(0) \in \mathsf{Coll}(\omega_1, <\kappa)$,
(2) $p(\gamma)$ is a countable partial function $f^p_\gamma : \kappa \to \omega_1$ such that

$$p \restriction \gamma \Vdash \text{“}f^p_\gamma \text{ is a partial specializing function of } \dot{T}_\gamma\text{”},$$

for every $\gamma > 0$,
(3) $p \restriction \gamma$ decides the tree order of $\dot{T}_\gamma$ on $\mathsf{dom}(f^p_\gamma)$,
(4) the support $\mathsf{sp}(p) := \{\gamma < \delta : p(\gamma) \neq \varnothing\}$ is countable.

The order is defined by pointwise inverse inclusion:

$$q \leqslant p \quad :\Longleftrightarrow \quad \forall \gamma < \delta \; q(\gamma) \supseteq p(\gamma).$$

Remark 2.3. For $\gamma < \delta$, there is a canonical complete embedding $\mathbb{P}_\gamma \hookrightarrow \mathbb{P}_\delta$ that takes a condition p in $\mathbb{P}_\gamma$ to $p^\frown(\varnothing, \varnothing, \dots)$ and each poset $\mathbb{P}_\delta$ is a dense subset of the iteration with iterands $(\dot{\mathbb{Q}}_\gamma : \gamma < \delta)$, where $\mathbb{Q}_0 = \mathsf{Coll}(\omega_1, <\kappa)$ and each $\dot{\mathbb{Q}}_\gamma$ is a name for the poset of countable approximations of specialising functions for $\dot{T}_\gamma$.

3. Traces

Notation 3.1. Let $\theta > \kappa^+$ be a regular cardinal and let $<_\theta$ be a well-ordering of H_θ. For each $\delta < \kappa^+$ and $\alpha \in \kappa$, let

$$M^\delta_\alpha := \text{Skolem hull of } \alpha \text{ in } (H_\theta, \in, <_\theta, \kappa, \delta).$$

Remark 3.2. We have $\omega_1 \cap M^\delta_\alpha = \alpha$ for club many $\alpha < \omega_1$.

Remark 3.3. We may assume without loss of generality that each M^δ_α contains the bookkeeping function $\gamma \to \dot{T}_\gamma$ as element.

Remark 3.4. $\mathbb{P}_\delta \in M^\delta_\alpha$.

Definition 3.5. Let $\delta < \kappa^+$. For each $p \in \mathbb{P}_\delta$ and $\alpha \in \kappa$, the **trace** $p|_\alpha$ of p into M^δ_α is a function on δ defined by

$$p|_\alpha(\gamma) := \begin{cases} p(\gamma) \restriction M^\delta_\alpha & \text{if } \gamma \in \delta \cap M^\delta_\alpha, \\ \varnothing & \text{otherwise.} \end{cases}$$

Remark 3.6. $p|_\alpha(0) \in \mathsf{Coll}(\omega_1, <\alpha)$ whenever $\omega_1 \cap M^\delta_\alpha = \alpha$.

Remark 3.7. In general, $p|_\alpha$ might not be a condition, but it will for stationarily many $\alpha \in \kappa$. The goal is to show that for stationarily many α, for every $p \in \mathbb{P}_\delta$, the trace $p|_\alpha$ is a *residue* for p, in the sense defined below in 1.1.

4. Preservation of κ

We show by induction on $\delta < \kappa^+$ that $\mathbb{P}_\delta$ has κ-cc and is strongly proper with respect to M^δ_α for $\mathcal{F}_{\mathsf{wc}}$-many $\alpha \in \kappa$. Here $\mathcal{F}_{\mathsf{wc}}$ is the **weakly compact filter** generated by sets

$$\{\alpha \in \kappa : V_\kappa \models \varphi(A) \Rightarrow V_\alpha \models \varphi(A \cap V_\alpha)\},$$

for a Π^1_1-formula $\varphi(X)$ and a set $A \subseteq V_\kappa$. The filter $\mathcal{F}_{\mathsf{wc}}$ is a normal filter on κ that extends the club filter. See Proposition 6.11. from Kanamori [2]. An important fact is that the statement "*T is a κ-Aronszajn tree*" is Π^1_1 for any $T \subseteq V_\kappa$, and furthermore, so is the statament

$$\text{“} \Vdash_{\mathbb{P}} \dot{T} \text{ is a } \kappa\text{-Aronszajn tree”},$$

whenever $\mathbb{P}, T \subseteq V_\kappa$.

The goal is to show the following proposition:

Proposition 4.1. *Let $\delta < \kappa^+$.*

(1) $\mathbb{P}_\delta$ has κ-cc.
(2) It holds for $\mathcal{F}_{\mathsf{wc}}$-many $\alpha < \kappa$ that if $p, q \in \mathbb{P}_\delta$ have the same trace to M^δ_α, then they have a common residue into M^δ_α.

The rest of the section is devoted to the prove Proposition 4.1. The proof is by induction on δ. **From now onwards until the end of the paper, we fix one $\boldsymbol{\delta < \kappa^+}$ and assume that the proposition holds for all $\boldsymbol{\gamma < \delta}$.** We begin by proving a series of lemmas.

4.1. Preliminary lemmas.

Remark 4.2. The poset $\mathbb{P}_\delta$ has size κ. We tacitly assume that each $\mathbb{P}_\delta$ is coded as a subset of V_κ. Up to choosing the $<_\theta$-least function, we may assume that each model M^δ_α knows about this coding. This will be important when using the Π^1_1-reflection of κ.

We say that a subposet $\mathbb{Q} \subseteq \mathbb{P}_\gamma$ **determines** a subset $S \subseteq T_\gamma$ if for all nodes $s, t \in S$ the set of conditions deciding the tree-order of $\dot{T}_\gamma$ between s and t is dense in $\mathbb{Q}$.

Lemma 4.3. *Assume that $\mathbb{P}_\gamma$ has κ-cc for all $\gamma < \delta$. Then, it holds for club many $\alpha \in \kappa$ that for every $\gamma \in \delta \cap M^\delta_\alpha$, the poset $\mathbb{P}_\gamma \cap V_\alpha$ determines the set $\mathsf{Lev}_{<\alpha}(\dot{T}_\gamma)$.*

Proof. Follows from strong inaccessibility of κ and κ-cc of each $\mathbb{P}_\gamma$. □

Lemma 4.4. *For $\mathcal{F}_{\mathsf{wc}}$ many $\alpha \in \kappa$, it holds that for every $p \in \mathbb{P}_\delta$, the trace $p|_\alpha$ is a condition in $\mathbb{P}_\delta \cap M^\delta_\alpha$.*

Proof. Follows from the induction hypothesis and the lemma above. □

Lemma 4.5. *If $p, q \in \mathbb{P}_\delta$ have a common residue r into M^δ_α, then there are $\hat{p} \leqslant p$ and $\hat{q} \leqslant q$ with a common trace into M^δ_α extending r.*

Lemma 4.6. *The set*

$$\{\alpha \in \kappa : \begin{array}{ll} (1) & V_\kappa \cap M^\delta_\alpha = V_\alpha, \\ (2) & \alpha \textit{ is an inaccessible cardinal}, \\ (3) & \forall \gamma \in \delta \cap M^\delta_\alpha \quad \mathbb{P}_\gamma \cap M^\gamma_\alpha \textit{ has } \alpha\textit{-cc}, \\ (4) & \forall \gamma \in \delta \cap M^\delta_\alpha \quad \mathbb{P}_\gamma \cap M^\gamma_\alpha \subseteq_c \mathbb{P}_\gamma, \\ (5) & \forall \gamma \in \delta \cap M^\delta_\alpha \quad \Vdash_{\mathbb{P}_\gamma \cap M^\gamma_\alpha} \text{“}\alpha = \aleph_2\text{”}, \\ (6) & \forall \gamma \in \delta \cap M^\delta_\alpha \quad \Vdash_{\mathbb{P}_\gamma \cap M^\gamma_\alpha} \text{“}\dot{T}_\gamma \cap V_\alpha \textit{ is an Aronszajn tree on } \alpha\text{”}, \\ (7) & \forall \gamma \in \delta \cap M^\delta_\alpha \quad \forall p \in \mathbb{P}_\gamma \ p|_\alpha \in \mathbb{P}_\gamma \cap M^\gamma_\alpha \} \end{array}$$

is in $\mathcal{F}_{\mathsf{wc}}$.

Proof. Points (1), (2) follow directly from the fact that κ is weakly compact, in particular regular and Mahlo. Point (3) follows by Π^1_1-reflection from the hypothesis that $\mathbb{P}_\gamma$ has κ-cc. Point (4) follows from (3) and normality of $\mathcal{F}_{\mathsf{wc}}$ by a pressing down argument. Points (5), (6) follow by Π^1_1-reflection. Point (7) is previous lemma. □

4.2. **Splitting.**

Definition 4.7. Let $p, q \in \mathbb{P}_\delta$, $\gamma < \delta$ and let $s, t \in T_\gamma$. We say that the conditions $\boldsymbol{p}$ **and** $\boldsymbol{q}$ **split the pair** $\boldsymbol{(s,t)}$ **below** $\boldsymbol{\alpha}$ if there are two distinct nodes $\hat{s}, \hat{t} \in T_\gamma \cap \alpha$ of the same height such that

$$p \upharpoonright \gamma \Vdash \hat{s} < s,$$
$$q \upharpoonright \gamma \Vdash \hat{t} < t.$$

The case $s = t$ is not excluded; we say that p and q **split the node** $\boldsymbol{s}$ **below** $\boldsymbol{\alpha}$ if p and q split the pair (s, s) below α.

The use of the weakly compact cardinal is substantial in the following lemma.

Lemma 4.8 (Splitting)**.** *The following holds for* $\mathcal{F}_{wc}$*-many* $\alpha < \kappa$*: for any two conditions* $p, q \in \mathbb{P}_\delta$ *with same trace to* M^δ_α *there are two conditions* $\hat{p} \leqslant p$ *and* $\hat{q} \leqslant q$ *that have the same trace to* M^γ_α *and that split below* α *every pair of nodes* $(s,t) \in (\mathsf{dom}(f^{\hat{p}}_\gamma) \times \mathsf{dom}(f^{\hat{q}}_\gamma)) - \alpha$ *where* $\gamma \in \delta \cap M^\delta_\alpha$.

Proof. We begin with a claim that we will then iterate to finish the proof of the lemma.

Claim 4.9. *The following holds for* $\mathcal{F}_{wc}$*-many* $\alpha < \kappa$*: for any* $\gamma \in \delta \cap M^\delta_\alpha$*, any two conditions* $p, q \in \mathbb{P}_\gamma$ *that have a common residue* r *into* M^γ_α*, and any nodes* $s, t \in T_\gamma - \alpha$ *there are two conditions* $\hat{p} \leqslant p$ *and* $\hat{q} \leqslant q$ *that have a common residue* $\hat{r} \leqslant r$ *into* M^γ_α *and that split the pair* (s,t) *below* α.

Proof of Claim 4.9. We show that the claim holds for every $\alpha < \kappa$ from the set in Lemma 4.6. Let $\gamma \in \delta \cap M^\delta_\alpha$ and suppose that $p, q \in \mathbb{P}_\gamma$ are conditions with same trace to M^γ_α. Let $s, t \in T_\gamma - \alpha$. The reflection properties of the weakly compact are essential in the following step. Let $G \subseteq \mathbb{P}_\gamma \cap M^\gamma_\alpha$ be a generic filter containing the common residue of p and q. It follows that p and q are in the quotient forcing $\mathbb{P}_\gamma / G$. By the choice of α, it holds (among other things) that

$$(*) \qquad \Vdash_{\mathbb{P}_\gamma \cap M^\gamma_\alpha} \text{“}\dot{T}_\gamma \cap \alpha \text{ is an } \alpha\text{-Aronszajn tree”}.$$

Thus $(\dot{T}_\gamma)_G$ is an α-Aronszajn tree, and thus the branch below the node s must be introduced by the quotient forcing $\mathbb{P}_\gamma/G$, for otherwise it would be a cofinal branch in $(\dot{T}_\gamma \cap \alpha)_G$. Thus, there must be two conditions $p^L, p^R \leqslant p$ in $\mathbb{P}_\gamma/G$ that split s at some level $\beta < \alpha$ with some distinct nodes s^L and s^R. Since p^L and p^R are in the quotient forcing $\mathbb{P}_\gamma/G$, there is some $r_0 \in G$ that forces $p, q \in \mathbb{P}_\gamma/\check{G}$. This r_0 is a common residue for p and q. Then we find an extension $\hat{q} \leqslant q$, also in the quotient $\mathbb{P}_\delta/G$, such that $\hat{q}$ decides the node below t at level β, call it $\bar{t}$. There is $r \in G$ that extends r_0 and forces $\hat{q} \in \mathbb{P}_\delta/\check{G}$. Then r is a common residue for p^L, p^R and $\hat{q}$. If $\bar{t} \neq s^L$, then p^L and $\hat{q}$ are as wanted, and if $\bar{t} \neq s^R$, then p^R and $\hat{q}$ are as wanted. This ends the proof of the claim. □

We now complete the proof of Lemma 4.8 by iterating Claim 4.9 countably many times. Using a suitable enumeration, we find conditions $(p_n)_{n<\omega}$ and $(q_n)_{n<\omega}$ such that conditions p_n and q_n have a common residue r_n into M^δ_α and split a pair of nodes (s_n, t_n) from $T_{\gamma_n} - \alpha$, where $\gamma_n \in \delta \cap M^\delta_\alpha$, $s_n \in \mathsf{dom}(f^{p_n}_{\gamma_n})$ and $t_n \in \mathsf{dom}(f^{q_n}_{\gamma_n})$.

This is accomplished by first fixing a bijection $\langle\cdot\rangle : \omega \times \omega \times \omega \to \omega$ that is suitable in the sense that whenever $n = \langle m, k, l\rangle$, then $m \leqslant n$. We begin by letting $p_0 := p$ and $q_0 := q$. At step $n+1$, we assume that p_n and q_n have been defined and have the same trace to M^δ_α. We first enumerate the union of the supports $(\mathsf{sp}(p_n) \cup \mathsf{sp}(q_n)) \cap M^\delta_\alpha$ as $(\gamma^n_k : k < \omega)$, and the set of all nodes in $(\mathsf{dom}(f^{p_n}_{\gamma^n_k}) - \alpha) \times (\mathsf{dom}(f^{q^n}_{\gamma^n_k}) - \alpha)$, for each $k < \omega$, as $((s_{(n,k,l)}, t_{(n,k,l)}) : l < \omega)$. Then we pick the unique m, k, l such that $n = \langle m, k, l\rangle$ and look at the conditions $p_n \upharpoonright \gamma^k_m$ and $q_n \upharpoonright \gamma^k_m$. We apply the induction hypothesis of Proposition 4.1(2): since $p_n \upharpoonright \gamma^k_m$ and $q_n \upharpoonright \gamma^k_m$ have the same trace to $M^{\gamma^k_m}_\alpha$, there is $r \in \mathbb{P}^k_{\gamma m} \cap M^{\gamma^k_m}_\alpha$ that is a common residue for them. Then we apply Claim 4.9 to the nodes $s_{(m,k,l)}$ and $t_{(m,k,l)}$: we find two conditions $p' \leqslant p_n \upharpoonright \gamma^k_m$ and $q' \leqslant q_n \upharpoonright \gamma^k_m$ in $\mathbb{P}_{\gamma^k_m}$ that split $s_{(m,k,l)}$ and $t_{(m,k,l)}$ and have a common residue $r' \leqslant r$ into $M^{\gamma^k_m}_\alpha$. By Lemma 4.5 we may extend further to assume that p' and q' have the same trace into $M^{\gamma^k_m}_\alpha$, and the same trace extends r'. We define p_{n+1} and q_{n+1} by taking the pointwise union:

$$p_{n+1} := p'^\frown p_n \upharpoonright [\gamma^k_m, \delta),$$
$$q_{n+1} := q'^\frown q_n \upharpoonright [\gamma^k_m, \delta).$$

Then p_{n+1} and q_{n+1} have the same trace to M^δ_α and split the nodes $s_{(m,k,l)}$ and $t_{(m,k,l)}$.

Finally, we let $\hat{p} := \bigcup_n p_n$ and $\hat{q} := \bigcup_n q_n$. They are as wanted.

□

4.3. **Proof of κ-cc and existence of common residues.**

We finally are ready to prove Proposition 4.1.

Proof of Proposition 4.1. The proof is by induction on δ. We assume that the proposition holds for $\gamma \in \delta$ and show that it holds for δ.

(1) We show that $\mathbb{P}_\delta$ has κ-cc. Let $\{p_\alpha : \alpha \in \kappa\} \subseteq \mathbb{P}_\delta$. We find distinct α and β such that p_α and p_β are compatible.

The general idea is as follows. By applying Splitting Lemma 4.8, for $\mathcal{F}_{\mathsf{wc}}$ many α, find conditions $p^L_\alpha, p^R_\alpha \leqslant p_\alpha$ which split relevant pairs of nodes and satisfy $p^L_\alpha|_\alpha = q^R_\alpha|_\alpha$. Then, by pressing down with Fodor's lemma, it is possible to show that for $\mathcal{F}_{\mathsf{wc}}$ many $\alpha < \beta$, the left extension p^L_α is compatible with the right extension p^R_β. This allows to finish.

Now in more detail. Look at the set $\mathcal{S}_\delta \in \mathcal{F}_{\mathsf{wc}}$ from Lemma 4.8. For every $\alpha \in \mathcal{S}_\delta$, find

$$p^L_\alpha, p^R_\alpha \leqslant p_\alpha$$

such that $p^L_\alpha|_\alpha = p^R_\alpha|_\alpha$ and for all $\gamma \in \delta \cap M^\delta_\alpha$, the pair (p^L_α, p^R_α) splits each pair of nodes (s,t) in $(\mathsf{dom}(f^{p^L_\alpha}_\gamma) \times \mathsf{dom}(f^{p^R_\alpha}_\gamma)) - V_\alpha$ with some pair of nodes $(\bar{s}, \bar{t})$ from V_α. By pigeonhole principle there is an $\mathcal{F}_{\mathsf{wc}}$-positive set $U \subseteq \mathcal{S}_\delta$ such that for all $\alpha, \beta \in U$ there are isomorphisms

$$p^L_\alpha \cong p^L_\beta,$$
$$p^R_\alpha \cong p^R_\beta,$$

which fix the traces

$$p^L_\alpha|_\alpha = p^L_\beta|_\beta = p^R_\beta|_\beta = p^R_\alpha|_\alpha.$$

Up to further refining U, we may assume that for all $\alpha < \beta$ from U and $\gamma \in \delta$,

$$\sup\{ht(t) : t \in (\mathsf{dom}(f^{p^L_\alpha}_\gamma) \cup \mathsf{dom}(f^{p^R_\alpha}_\gamma)) - V_\alpha\}$$
$$< \min\{ht(t) : t \in (\mathsf{dom}(f^{p^L_\beta}_\gamma) \cup \mathsf{dom}(f^{p^R_\beta})) - V_\beta\},$$

and that

$$(\mathsf{sp}(p^L_\alpha) - \alpha) \cap (\mathsf{sp}(p^R_\beta) - \beta) = \varnothing.$$

Choose $\alpha < \beta$ from U. We claim that $p^L_\alpha \| p^R_\beta$. Define $\hat{p}$ to be the pointwise union, by letting

$$\hat{p}(0) := p^L_\alpha(0) \cup p^R_\beta(0),$$
$$\hat{p}(\gamma) := f^{p^L_\alpha}_\gamma \cup f^{p^R_\beta}_\gamma.$$

We claim that $\hat{p}$ extends to a condition. We need to make sure that $\hat{p} \restriction \gamma$ decides the tree-order of $\dot{T}_\gamma$ relativised to $\mathsf{dom}(f^{\hat{p}}_\gamma)$ and that show that $f^{\hat{p}}_\gamma$ is forced to be a specializing function. If to the contrary this is not the case, then there are two distinct nodes $s, t \in \mathsf{dom}(f^{\hat{p}}_\gamma)$ such that

$$f^{\hat{p}}_\gamma(s) = f^{\hat{p}}_\gamma(t)$$

and such that $\hat{p} \restriction \gamma \Vdash s \leqslant t$. If $s, t \in \mathsf{dom}(f^{p^L_\alpha}_\gamma)$ or $s, t \in \mathsf{dom}(f^{p^R_\beta}_\gamma)$, we are done. Assume thus that $s \in \mathsf{dom}(f^{p^L_\alpha}_\gamma)$ and $t \in \mathsf{dom}(f^{p^R_\beta}_\gamma)$. Now, by construction,

$$p^L_\alpha \restriction \gamma \Vdash \bar{s} < s,$$
$$p^R_\beta \restriction \gamma \Vdash \bar{t} < t.$$

This is because the pair $(p^L_\alpha \restriction \gamma, p^R_\alpha \restriction \gamma)$ splits some pair (s, t') with $(\bar{s}, \bar{t}')$ and the pair $(p^L_\beta \restriction \gamma, p^R_\beta \restriction \gamma)$ splits a pair (s', t) with $(\bar{s}', \bar{t})$. This implies

$$\hat{p} \Vdash s \perp t.$$

(2) We show that for $\mathcal{F}_{\mathsf{wc}}$ many $\alpha \in \kappa$, whenever p and q are two conditions in $\mathbb{P}_\delta$ with the same trace $p|_\alpha = q|_\alpha$ to M^δ_α, then they have a common residue to M^δ_α.

- Suppose to the contrary that there is an $\mathcal{F}_{\mathsf{wc}}$-positive set S such that for every $\alpha \in S$, there are two conditions p_α and q_α with the same trace $p_\alpha|_\alpha = q_\alpha|_\alpha$ and such that no condition in $\mathbb{P}_\delta \cap M^\delta_\alpha$ is a common residue for them. We may choose the set S such that every $\alpha \in S$ satisfies all properties listed in Lemmas 4.3, 4.4 and 4.6. Furthermore, we may assume by the splitting lemma 4.8 that for every $\alpha \in S$, there are $\hat{p}_\alpha \leqslant p_\alpha$ and $\hat{q}_\alpha \leqslant q_\alpha$ with same traces $\hat{p}_\alpha|_\alpha = \hat{q}_\alpha|_\alpha$ and such that all relevant pairs of nodes are split.
- By stratifying by levels we may arrange that for $\alpha < \beta$ in S', the conditions $\hat{p}_\alpha$ and $\hat{q}_\beta$ are compatible, as in the proof of κ-cc.
- For every $\alpha \in S$ there is a maximal antichain

  $W_\alpha \subseteq \{r \in \mathbb{P}_\delta \cap M^\delta_\alpha : r \leqslant \hat{p}_\alpha|_\alpha = \hat{q}_\alpha|_\alpha \text{ and } (r \perp \hat{p}_\alpha \text{ or } r \perp \hat{q}_\alpha)\}.$
- Since we already proved κ-cc of $\mathbb{P}_\delta$, we may assume that $\mathbb{P}_\delta \cap M^\delta_\alpha \subseteq_c \mathbb{P}_\delta / \hat{p}_\alpha$ for every $\alpha \in S$. This implies that W_α is a maximal antichain in $\mathbb{P}_\delta / \hat{p}_\alpha|_\alpha$.
- Each antichain W_α has size $< \kappa$. There is a subset $S' \subseteq S$ such that $W_\alpha = W_\beta$ for $\alpha, \beta \in S'$, and such that for $r \in W$, $\hat{p}_\alpha \perp r$ iff $\hat{p}_\beta \perp r$ and $\hat{q}_\alpha \perp r$ iff $\hat{q}_\beta \perp r$. This follows by Fodor's lemma applied to the fact that wlog we may fix an enumeration of all antichains $\mathbb{P}_\delta$ of length κ and since there are stationarily many inaccesibles below κ, there are many $\alpha < \kappa$ such that the enumeration gvies an enumeration of antichains of $\mathbb{P}_\delta \cap M^\delta_\alpha$ of length α; then the function mapping α to the index of W_α is regressive.
- Choose some $v \leqslant \hat{p}_\alpha, \hat{q}_\beta$. Then $v \leqslant \hat{p}_\alpha|_\alpha$ and without loss of generality v does not belong to $\mathbb{P}_\delta \cap M^\delta_\alpha$. Thus $v \notin W_\alpha = W$ either.
- We claim that $W \cup \{v\}$ is an antichain. This will contradict the maximality of W. Indeed, if $w \in W$, then either $w \perp \hat{p}_\alpha$ or $w \perp \hat{q}_\beta$. The first case implies that $w \perp v$ and the second case implies that $w \perp v$. Thus $W \cup \{v\}$ is indeed an antichain. This is a contradiction.

□

Corollary 4.10. *Each poset $\mathbb{P}_\delta$ is κ-strongly proper and have κ-cc, for $\delta < \kappa^+$. The poset $\mathbb{P}_{\kappa^+}$ has κ-cc.*

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