AROUND THE INFINITARY LOGIC $\mathcal{L}^1_\kappa$

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Abstract

The logic $\mathcal{L}^1_\kappa$ provides a positive answer to the question whether there are logics, beyond the first order logic, that simultaneously admit a Lindström style characterization and have the interpolation property. This text presents the definition of the logic $\mathcal{L}^1_\kappa$ and its two characterization theorems, which form the main results of the article [11] by S. Shelah.

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1 Introduction

The main objects of study in abstract model theory are model classes. A model class is a class of structures in one fixed signature, closed under isomorphism. A model class is *definable* in a logic $\mathcal{L}$ if there is a sentence in $\mathcal{L}$ in the signature of the model class, such that the models of that sentence are exactly the models of that model class. A question to ask is: how to design a logic which is powerful enough to define a maximal number of model classes, yet tame enough to admit useful mode-theoretic properties.

For an abstract model theorist, therefore, a logic is not primarily a deductive system, but a formal language. Indeed, in abstract model theory, the essence of a sentence is the model class it defines. The main purpose of a logic is to express mathematical concepts in an unambiguous and exact manner. This is somewhat far from a proof-theorist's view, namely, that the deductive rules are those that constitute a logic.

The main subject of this text is the logic $\mathcal{L}^1_\kappa$. It was discovered by S. Shelah in 2012, in [11], and it provides a positive solution to the problem whether there are logics beyond the first order logic that admit a Lindström style characterization and have the interpolation property. These properties will be defined later. The logic $\mathcal{L}^1_\kappa$ is an infinitary logic, between the logics $\mathcal{L}_{\kappa\omega}$ and $\mathcal{L}_{\kappa\kappa}$. The logic $\mathcal{L}_{\kappa\omega}$ is defined by allowing conjunctions of length $< \kappa$ and the logic $\mathcal{L}_{\kappa\kappa}$ is defined by allowing, in addition, quantification over tuples of length $< \kappa$. Here κ is always assumed to be an uncountable cardinal, usually a fixed point of the beth function.

Most of the work that have been done to obtain better picture of $\mathcal{L}^1_\kappa$ has been done in the last ten years, by Shelah himself in [11] and [12], Villaveces and Väänänen [17], Dzamonja and Väänänen [4], Veličković and Väänänen [15] and Shelah and Villaveces [13].

As opposed to the standard extensions of the first order logic (such as $\mathcal{L}_{\kappa\omega}$ and $\mathcal{L}_{\kappa\kappa}$) the logic $\mathcal{L}^1_\kappa$ is not defined by enriching the first-order syntax. Instead, it is defined purely semantically. The sentences are obtained by imposing an equivalence relation on the class of structures and taking a sentence to be a (union of a) collection of equivalence classes. Therefore, in this logic, a sentence is a model class by definition, contrary to being a string of symbols.

In fact, it is not known whether $\mathcal{L}^1_\kappa$ admits a (recursive) syntax. This may seem as a very serious defect, and one may raise the question whether a logic with no recursive syntax should be counted as a logic at all. However, despite not being defined from the syntax, $\mathcal{L}^1_\kappa$ has some desirable model-theoretic properties.

The most striking property is the interpolation property. Interpolation is a strong property, that allows one to eliminate "useless symbols", in the following sense: if φ is a sentence that implies another sentence φ', possibly in a different signature, then there is a sentence ψ in the common signature, such that φ implies ψ and ψ implies φ'. For example, in the case of the logic $\mathcal{L}^1_\kappa$, interpolation has the consequence that all the cardinalities below κ are definable in the empty signature, which is not the case, for example, in the logic $\mathcal{L}_{\kappa\omega}$.

The main reason, why it is such a remarkable thing for $\mathcal{L}^1_\kappa$ to have interpolation, is that the interpolation is an extremely rare property among the abstract logics. For instance, all infinitary logics of the standard form $\mathcal{L}_{\kappa\lambda}$, in which κ is at least ω_2, miserably fail it.

In addition to interpolation, the logic $\mathcal{L}^1_\kappa$ admits a Lindström style characterization theorem. A Lindström style characterization of an abstract logic is a characterization of a logic as a maximal logic with respect to some properties, usually compactness properties.

The original Lindström theorem was discovered in the late sixties, by the Swedish logician P. Lindström, in [9]. The proof was for the first order logic, and it states that the first order logic is a maximal logic which is countably compact and satisfies the Löwenheim-Skolem theorem. Lindström's theorem means that if one wants to increase the expressive power of the first order logic, one must be ready to let go one of these two properties. Variations of the proof give the same theorem with respect to other combinations of properties, such as compactness and the Tarski Union property[1] [10], or Löwenheim-Skolem theorem and well-ordering number[2] ω, see [1]. Väänänen wrote in [14] about the first order version: *"[Lindström's theorem] says in effect that any formal language that goes beyond first order logic has to distinguish between some infinite cardinalities in the sense that some sentence has a model of some infinite cardinality but not of all infinite cardinalities."*.

In fact, in the case of the first order logic, the proof of the Lindström theorem yields an another proof of the interpolation property. This led to the question whether the Löwenheim-Skolem theorem could be replaced by interpolation, in the formulation of the Lindström theorem. However, this question is still open.

The main theorem of this text is the Lindström theorem for the logic $\mathcal{L}^1_\kappa$, due to S. Shelah in [11]. It is a characterization of $\mathcal{L}^1_\kappa$ as the maximal logic having a certain weak compactness property, called *strong well-ordering number*. Alternatively, analogously to the first-order version, also this characterization theorem can be formulated in terms of a weak form of a Tarski Union property and a weak form of a Löwenheim-Skolem property. This theorem is proved in the fourth chapter, after presenting the necessary preliminaries in the second and the third chapter.

Shelah also proved in [11] another characterization theorem for $\mathcal{L}^1_\kappa$. Namely, $\mathcal{L}^1_\kappa$ is the minimal logic able to define a certain covering property, which has a weak form of interpolation. Together, this minimality result and the Lindström type maximality result give a full characterization of $\mathcal{L}^1_\kappa$. This is beautiful, especially because it is a characterization that makes use of (a weak form of) interpolation. As mentioned, it is an open question whether the first order logic could be characterized in terms of (full) interpolation. This second characterization theorem is presented in the fifth chapter.

[1] A logic satisfies the *Tarski Union property* if every structure of an elementary chain is an elementary submodel of the union of the chain

[2] The *well-ordering number* of a logic is the least ordinal α such that every sentence that has a model which is an extended expansion of $(\alpha, \in)$, also has a model $\mathcal{B}$ in which $(\alpha^{\mathcal{B}}, \in^{\mathcal{B}})$ is ill-founded. Having a well ordering number is a weak form of compactness property.

Notation

The symbol $\in$ is preserved for the set-theoretic membership relation. The letter n denotes a natural number, the greek lettes $\alpha, \beta, \gamma, \ldots$ denote ordinals and the letters κ and λ denote (infinite) cardinals, if it is not indicated otherwise. Tuples are denoted by a bar on the top and they can be finite or infinite. The concatenation is denoted by $\bar{x}^\frown\bar{y}$ or $\bar{x}^\frown y$, if y is an element. If A is a set and θ is a cardinal, then $[A]^{\leqslant\theta}$ denotes the collection of the subsets of A of size $\leqslant \theta$. If A and B are sets, then A^B denotes the set of functions from B into A, i.e. the tuples of elements of A indexed by elements of the set B. For a cardinal λ, the set $H(\lambda)$ or H_λ is the set of sets of hereditary cardinality less than λ. The *beth-function* is defined as follows

$$\begin{aligned}
&\beth_0 := \omega.\\
&\beth_{\alpha+1} := 2^{\beth_\alpha}.\\
&\beth_\delta := \bigcup_{\alpha<\delta} \beth_\alpha, \quad \text{for a limit ordinal } \delta.
\end{aligned}$$

All logics considered in this text will live in the same signatures and structures as the first-order logic. A *signature* is a set τ containing constant symbols, function symbols and relation symbols, each of finite arity. Constant symbols might be identified with "zero-ary" function symbols. An *isomorphism of signatures* is a bijection that respects arities. For a signature τ, a τ*-structure* $\mathcal{M}$ is a set M together with an interpretation $s^{\mathcal{M}} \subseteq M^n$ for each n-ary symbol s from τ. Write $Str(\tau)$ for the class of all τ-structures. Sometimes the word *model* is used synonymously with the word structure. In general, structures are denoted by calligraphic letters $\mathcal{M}$, and their underlying sets, their *domains*, are either confused with the structure $\mathcal{M}$, by abusing notation, or denoted by ordinary capital letters M. For a signature τ, two τ-structures are *isomorphic* if there is a bijection between them that preserves the structure induced by τ. A structure $\mathcal{M}$ is an *expansion* of a τ-structure if $\mathcal{M}$ is a σ-structure for some signature $\sigma \supseteq \tau$. In that case, the restriction of $\mathcal{M}$ to the signature τ is called a τ*-reduct* and denoted by $\mathcal{M} \restriction \tau$ or explicitly $(M, \tau^{\mathcal{M}}) := (M, s^{\mathcal{M}})_{s\in\tau}$.

If $\mathcal{M}$ is a structure in some signature τ and $a \in \mathcal{M}$ is an element, then $(\mathcal{M}, a)$ denotes the $\tau \cup \{\dot{a}\}$-structure, where $\dot{a}$ is a new constant symbol, with interpretation a. Similarly, if $R \subseteq [\mathcal{M}]^n$, for some integer n, then $(\mathcal{M}, R)$ denotes the $\tau \cup \{\dot{R}\}$-expansion of $\mathcal{M}$, where $\dot{R}$ is an n-ary relation symbol interpreted as R. The same thing can be done with any function.

For any signature τ, a τ*-formula in variables* $\bar{x} = (x_i)_{i\in I}$ is a sentence in signature $\tau \cup \{c_i\}_{i\in I}$, for new constant symbols c_i, $i \in I$. Let $\mathcal{M}$ be a τ-structure and let $\bar{a} = (a_i)_{i\in I} \in \mathcal{M}^I$. We write $\mathcal{M} \models \varphi(\bar{a})$ for $(\mathcal{M}, \bar{a}) \models \varphi$, where $(\mathcal{M}, \bar{a})$ is the $\tau \cup \{c_i\}_{i\in I}$-structure obtained by expanding $\mathcal{M}$ by interpreting each c_i by a_i.

2 Abstract logics

The next definition of a *logic* is originally due to P. Lindström, [9], and this version follows [1]. Sometimes a logic is called an *abstract logic*, in order to emphasize the distinction from the non-technical term.

Typical examples of abstract logics are the infinitary logics $\mathcal{L}_{\kappa\lambda}$, which are defined just as the first-order logic, but allowing infinitary conjunctions and quantifiers over infinite tuples. Other examples include the higher order logics and the logics with generalized quantifiers.

A *signature* τ is always a set of constant, function and relation symbols, each of finite arity.

Definition 2.1. A *logic* or an *abstract logic* is a class function on signatures τ,

$$\tau \mapsto (\mathcal{L}(\tau), \models_{\mathcal{L}(\tau)}),$$

where $\mathcal{L}(\tau)$ is the *collection of τ-sentences* and $\models_{\mathcal{L}(\tau)}$ is a relation between the τ-structures and τ-sentences, abbreviated by $\models$, if no ambiguities arise. In addition, the following must be satisfied:

1. (Isomorphism Property). For every signature τ and for all τ-structures $\mathcal{M}$ and $\mathcal{N}$, if $\mathcal{M}$ and $\mathcal{N}$ are isomorphic, then

$$\text{for all } \varphi \in \mathcal{L}(\tau), \quad \mathcal{M} \models \varphi \iff \mathcal{N} \models \varphi.$$

2. (Renaming property). Every isomorphism of signatures $\tau \to \tau^*$, $s \mapsto s^*$ induces an isomorphism $\mathcal{L}(\tau) \to \mathcal{L}(\tau^*)$, $\varphi \mapsto \varphi^*$ such that for every $\mathcal{M} \in Str(\tau)$,

$$\mathcal{M} \models \varphi \iff \mathcal{M}^* \models \varphi^*,$$

where $\mathcal{M}^*$ is the τ^*-structure obtained from $\mathcal{M}$ by taking the same domain and interpreting $s^{*\mathcal{M}^*} = s^{\mathcal{M}}$, for all symbols $s \in \tau$.

3. (Monotonicity). For all signatures τ and σ, if $\tau \subseteq \sigma$, then $\mathcal{L}(\tau) \subseteq \mathcal{L}(\sigma)$.

4. (Expansion property). If $\varphi \in \mathcal{L}(\tau)$ and $\mathcal{M}$ is an expansion of a τ-structure, then

$$\mathcal{M} \models \varphi \iff \mathcal{M} \restriction \tau \models \varphi.$$

Let $\mathcal{L}$ be a logic and let τ be a signature. Let $\Sigma \subseteq \mathcal{L}(\tau)$ be a collection of τ-sentences. Say that two τ-structures $\mathcal{M}$ and $\mathcal{N}$ *agree on* Σ, $\mathcal{M} \equiv_\Sigma \mathcal{N}$, if for all sentences $\varphi \in \Sigma$,

$$\mathcal{M} \models_{\mathcal{L}} \varphi \iff \mathcal{N} \models_{\mathcal{L}} \varphi.$$

In case $\Sigma = \mathcal{L}(\tau)$, we write $\mathcal{M} \equiv_{\mathcal{L}} \mathcal{N}$ and say that $\mathcal{M}$ and $\mathcal{N}$ are *$\mathcal{L}$-elementarily equivalent.*

For a logic $\mathcal{L}$, a signature τ and a sentence $\varphi \in \mathcal{L}(\tau)$, the class of the *models of* φ is defined as

$$Mod_{\mathcal{L},\tau}(\varphi) := \{\mathcal{M} \in Str(\tau) \ : \ \mathcal{M} \models_{\mathcal{L}(\tau)} \varphi\}.$$

We may drop the subscript if the logic and the signature are clear from the context and write simply $Mod(\varphi)$.

A *model class* is a class of structures in one fixed signature, closed under isomorphism. A model class $\mathcal{K}$ is *definable in a logic* $\mathcal{L}$ if there is an $\mathcal{L}$-sentence φ in the signature of $\mathcal{K}$, such that

$$\mathcal{K} = Mod_{\mathcal{L},\tau}(\varphi).$$

Definition 2.2. Let $\mathcal{L}$ and $\mathcal{L}'$ be logics. The logic $\mathcal{L}'$ is *stronger than* $\mathcal{L}$, $\mathcal{L}' \geqslant \mathcal{L}$, if every model class definable in $\mathcal{L}$ is also definable in $\mathcal{L}'$. Two logics are *equivalent*, $\mathcal{L} \equiv \mathcal{L}'$, in case $\mathcal{L} \leqslant \mathcal{L}'$ and $\mathcal{L}' \leqslant \mathcal{L}$.

The phrases "$\mathcal{L}$ is a *sublogic* of $\mathcal{L}'$, "$\mathcal{L}$ is *below* $\mathcal{L}'$, "$\mathcal{L}'$ is *above* $\mathcal{L}$ are used synonymyously with "$\mathcal{L}'$ is stronger than $\mathcal{L}$".

The expressive power of a logic is thus measured purely semantically. Roughly: the more a logic admits definable model classes, the stronger it is.

Example: The infinitary logics $\mathcal{L}_{\kappa\lambda}$ and well-orders

Definition 2.3 (The logic $\mathcal{L}_{\kappa\lambda}$)**.** For regular cardinals κ and λ, the logic $\mathcal{L}_{\kappa\lambda}$ is defined as follows: For a signature τ, the set of τ-sentences $\mathcal{L}_{\kappa\lambda}(\tau)$ is obtained by closing the set of atomic τ-formulas under the first-order operations $\neg, \wedge$ and $\exists$ together with the infinitary conjunctions $\bigwedge_\alpha$, for $\alpha < \kappa$, and infinitary quantifiers $\exists \bar{x}$, for $|\bar{x}| < \lambda$. In addition, it is required that in each formula, only less than λ variables occur. The semantic is defined as in the first-order case, with the additional inductive clauses

$$\begin{aligned}
\mathcal{M} \models \bigwedge_i \varphi_i(\bar{a}_i) \quad &:\iff \quad \text{for each } i, \quad \mathcal{M} \models \varphi_i(\bar{a}_i). \\
\mathcal{M} \models \exists \bar{x} \varphi(\bar{x}, \bar{a}) \quad &:\iff \quad \text{there is } \bar{b} \in \mathcal{M}^{|\bar{x}|} \text{ such that } \mathcal{M} \models \varphi(\bar{b}, \bar{a}).
\end{aligned}$$

A logic $\mathcal{L}$ is strong enough *define well-order*, if there is an $\mathcal{L}(\{<\})$-sentence which defines the class of well-orders. This class is naturally defined of course in the second order logic, but it is easy to define also in the logic $\mathcal{L}_{\omega_1\omega_1}$. The axioms of linear order are expressible in first-order and hence also in $\mathcal{L}_{\omega_1\omega_1}$, and furthermore, there is an $\mathcal{L}_{\omega_1\omega}$-sentence that expresses "*there is no infinite descending chain*". This sentence is

$$\neg \exists \bar{x} \bigwedge_{i<\omega} x_{i+1} < x_i,$$

where $\bar{x} = (x_i)_{i<\omega}$. The first-order logic is not strong enough to define well-order. Neither is $\mathcal{L}_{\kappa\omega}$, for any cardinal κ. However, for each ordinal $\beta < \kappa$ separately, we will see that there is an $\mathcal{L}_{\kappa\omega}(\{<\})$-sentence that defines the class of those well-orders that have order type at most β.

Definition 2.4. The *well-ordering number* of a logic, if exists, is the least ordinal γ such that every sentence that has a $\{G, <, \dots\}$-model $\mathcal{M}$, for a unary predicate symbol G and a binary predicate symbol $<$, in which $(G^{\mathcal{M}}, <^{\mathcal{M}})$ is isomorphic to $(\gamma, \in)$, also has a model $\mathcal{B}$ in which $(G^{\mathcal{B}}, <^{\mathcal{B}})$ is ill-founded.

The well-ordering number of the first-order logic is ω, because the compactness theorem of the first order logic allows to construct models with non-standard integers. We will see that the well-ordering number of $\mathcal{L}_{\kappa\omega}$ is κ. The fact that it is at most κ will later follow from the Corollary 4.4. Let us see now why it is at least κ, by constructing for each ordinal $\beta < \kappa$ an $\mathcal{L}_{\kappa\omega}$-sentence that characterizes the structure $(\beta, \in)$ up to isomorphism. Here the relation $\in$ is always the set-theoretic natural order of the ordinal β.

Observation 2.5. *Let κ be an infinite cardinal. For every ordinal $\beta < \kappa$, there is an $\mathcal{L}_{\kappa\omega}(\{<\})$-formula* $\mathrm{rank}_\beta(x)$ *such that for all $\{<\}$-structures $\mathcal{A}$ and for all elements $a \in \mathcal{A}$, if*

$$\mathcal{A} \models \mathrm{rank}_\beta(a),$$

then the $<$-predecessors of the element a form a well-ordering of order-type β.

Proof. Define the formula by recursion as follows.

$$\begin{aligned}
\mathrm{rank}_0(x) \;&:\equiv\; \neg\exists y (y < x) \\
\mathrm{rank}_\beta(x) \;&:\equiv\; \forall y < x \bigvee_{\alpha<\beta} \mathrm{rank}_\alpha(y) \quad \wedge \quad \bigwedge_{\alpha<\beta} \exists! z \ (z < x \ \wedge \ \mathrm{rank}_\alpha(z)) \\
&\qquad \wedge \quad \textit{axioms of linear order relativized to the predecessors of } x.
\end{aligned}$$

Then $\mathrm{rank}_\beta(x)$ is as wanted. $\square$

Observation 2.6. *Let κ be an infinite cardinal. For every ordinal $\beta < \kappa$, there is an $\mathcal{L}_{\kappa\omega}(\{<\})$-sentence that characterizes $(\beta, \in)$ up to isomorphism.*

Proof. The sentence

$$\forall x \bigvee_{\alpha\in\beta} \mathrm{rank}_\alpha(x) \quad \wedge \quad \bigwedge_{\alpha<\beta} \exists! z \ \mathrm{rank}_\alpha(z) \quad \wedge \quad \textit{axioms of linear order.}$$

characterizes $(\beta, \in)$ up to isomorphism. $\square$

In fact, the logic $\mathcal{L}_{\kappa\omega}$ is powerful enough to characterize up to isomorphism also every expansion of a well-order of type $< \kappa$, which has a signature of size $< \kappa$.

Remark 2.7. Assume that τ is smaller than κ. For every structure of the form $\mathcal{A} = (A, <^{\mathcal{A}}, \tau^{\mathcal{A}})$, where $(A, <^{\mathcal{A}})$ is isomorphic to some substructure of $(\alpha, \in)$, for some ordinal $\alpha < \kappa$, there is an $\mathcal{L}_{\kappa\omega}(\tau \cup \{<\})$-sentence that characterizes $\mathcal{A}$ up to isomorphism.

Proof. We may assume that the domain of $\mathcal{A}$ is some ordinal, and without loss of generality we may also assume that $A = \alpha$, since otherwise we can collapse A and characterize the collapse.

- For every constant symbol $c \in \tau$, define

$$\chi_c \; :\equiv \; \mathrm{rank}_{c^{\mathcal{A}}}(c).$$

- For every k-ary function symbol $f \in \tau$, define

$$\chi_f \; :\equiv \; \forall \bar{x} \bigwedge_{\bar{\alpha} \in \beta^k} \left(\bigwedge_{i<k} \mathrm{rank}_{\alpha_i}(x_i) \to \mathrm{rank}_{f^{\mathcal{A}}(\bar{\alpha})}(f(\bar{x})) \right).$$

- For every k-ary relation symbol R, define

$$\chi_R \; :\equiv \; \forall \bar{x} \bigwedge_{\bar{\alpha} \in \beta^k} \left(\bigwedge_{i<k} \mathrm{rank}_{\alpha_i}(x_i) \leftrightarrow R^{\bar{\alpha}}(\bar{x}) \right),$$

  where

$$R^{\bar{\alpha}}(\bar{x}) = \begin{cases} R(\bar{x}) & \text{if } \bar{\alpha} \in R^{\mathcal{A}} \\ \neg R(\bar{x}) & \text{otherwise.} \end{cases}$$

By the Observation 2.6, there is a sentence χ_α that characterizes $(A, <)$, which is isomorphic to $(\alpha, \in)$, up to isomorphism. Then the sentence

$$\chi_\alpha \wedge \bigwedge_{s \in \tau} \chi_s$$

characterizes $\mathcal{A}$ up to isomorphism. $\square$

Regular logics

Atomic substitution

In any logic, it is important to be able to code many small structures inside a large structure, in some suitably modified signature, in such a manner that the large structure can talk about the satisfaction relation of the small ones.

Let τ be a signature and let P be some $(n+1)$-ary predicate symbol, not from τ. Define the signature τ^n by

- replacing every constant symbol c from τ by an n-ary function symbol c^n,
- replacing every k-ary function symbol f from τ by an $(n+k)$-ary function symbol f^n, and
- replacing every k-ary relation symbol R from τ by an $(n+k)$-ary relation symbol R^n.

Let $\mathcal{A}$ be an arbitrary $\tau^n \cup \{P, \dots\}$-structure. For a tuple $\bar{a} \in \mathcal{A}^n$, denote

$$P_{\bar{a}} = \{b : P^{\mathcal{A}}(\bar{a}, b)\}.$$

We define a τ-structure with domain $P_{\bar{a}}$. Interpret each symbol $c, f, R \in \tau$ in the set $P_{\bar{a}}$ as follows:

$$\begin{aligned} c^{P_{\bar{a}}} &:= c^n(\bar{a}) \\ f^{P_{\bar{a}}}(\bar{b}) &:= f^n(\bar{a}, \bar{b}) \\ R^{P_{\bar{a}}}(\bar{b}) &:\iff R^n(\bar{a}, \bar{b}). \end{aligned}$$

In case the set $P_{\bar{a}}$ contains each constant and is closed under each function under the above interpretations, say that it is $\tau_{\bar{a}}$-closed, and in this case, denote the resulting τ-structure by $(P_{\bar{a}}, \tau_{\bar{a}})$, or $(P_{\bar{a}}^{\mathcal{A}}, \tau_{\bar{a}}^{\mathcal{A}})$, if there is a risk of misunderstanding the ambient structure.

For a signature τ, a *τ-formula* is a sentence in some signature that extends τ with a set of new constant symbols. We call these new constants *variables*. If $\varphi(\bar{x})$ is a τ-formula in variables $\bar{x} = (x_i)_{i \in I}$, $\mathcal{M}$ is a τ-structure and $\bar{a} = (a_i)_{i \in I} \in \mathcal{M}^I$, then we write

$$\mathcal{M} \models \varphi(\bar{a}) \quad :\iff \quad (\mathcal{M}, \bar{a}) \models \varphi(\bar{x}).$$

Here $(\mathcal{M}, \bar{a})$ is the expansion of $\mathcal{M}$ into a $\tau \cup \{x_i\}_{i \in I}$-sructure via the interpretation $x_i \mapsto a_i$ for each $i \in I$.

A logic $\mathcal{L}$ has the *atomic substitution property* if the following holds for every signature τ:

For every sentence $\varphi \in \mathcal{L}(\tau)$ and every $(n+1)$-ary predicate symbol P not from τ, there is a formula $\varphi_{\bar{x}}^P := \varphi^P(\bar{x})$ in variables $\bar{x} = (x_i)_{i \in n}$ in the signature $\tau^n \cup \{P\}$, where τ^n is the signature obtained from τ by replacing each k-ary symbol by an $(n+k)$-ary symbol, such that for all $\tau^n \cup \{P, \dots\}$-structures $\mathcal{A}$ and tuples $\bar{a} \in \mathcal{A}^n$, the following are equivalent:

1. $\mathcal{A} \models \varphi_{\bar{a}}^P$.
2. If $P_{\bar{a}}$ is $\tau_{\bar{a}}$-closed, then $(P_{\bar{a}}, \tau_{\bar{a}}) \models \varphi$.

In a logic that satisfies the atomic substitution property, it is thus possible to relativize each sentence to a predicate, in the above sense.

Example: The logic $\mathcal{L}_{\kappa\omega}$ satisfies the atomic substitution property.

Let τ be a signature and let P be some $(n+1)$-ary predicate symbol, not from τ. Let τ^n be the signature obtained from τ by adding n to the arity of each symbol in τ. For each symbol $s \in \tau$, denote by s^n the corresponding symbol in τ^n.

Let $\bar{x} = (x_i)_{i \in n}$ be a tuple of variables. For a τ-term t, denote $t_{\bar{x}}$ the τ^n-term defined recursively as:

- $y_{\bar{x}} := \bar{x}^\frown y$.
- $c_{\bar{x}} := c^n(\bar{x})$.
- $f_{\bar{x}}(\bar{t}) := f^n(\bar{x}, \bar{t}_{\bar{x}})$.

Then define the formulas $\varphi^P_{\bar{x}}$ recursively as:

- If φ is $t = t'$, then $\varphi^P_{\bar{x}}$ is $t_{\bar{x}} = t'_{\bar{x}}$.
- If φ is $R(\bar{t})$, then $\varphi^P_{\bar{x}}$ is $R^n(\bar{x}, \bar{t}_{\bar{x}})$.
- If φ is $\neg\psi$ or $\bigwedge \psi_i$, then $\varphi^P_{\bar{x}}$ is $\neg\psi^P_{\bar{x}}$ or $\bigwedge(\psi_i)^P_{\bar{x}}$, respectively.
- If φ is $\exists y\psi$, then $\varphi^P_{\bar{x}}$ is $\exists y\ (\ P_{\bar{x}}(y)\ \wedge\ \psi^P_{\bar{x}})$.

Now, for any $\tau^n \cup \{P, \dots\}$-structure $\mathcal{A}$ and $\bar{a} \in \mathcal{A}^n$, if $P_{\bar{a}}$ is $\tau_{\bar{a}}$-closed, then

$$\mathcal{A} \models \varphi^P_{\bar{a}} \iff (P_{\bar{a}}, \tau_{\bar{a}}) \models \varphi,$$

as wanted. This shows that the logic $\mathcal{L}_{\kappa\omega}$ has atomic substitution.

Regularity properties

A logic $\mathcal{L}$ is *closed under first-order operations* if it satisfies the following.

1. (Atomic sentences). For every signature τ and atomic τ-sentence φ, there is a sentence $\psi \in \mathcal{L}(\tau)$ such that

$$Mod_{\mathcal{L},\tau}(\psi) = Mod_{\mathcal{L}_{\omega\omega},\tau}(\varphi).$$

2. (Conjunction). For every signature τ and any sentences $\varphi, \psi \in \mathcal{L}(\tau)$, there is a sentence $\varphi \wedge \psi \in \mathcal{L}(\tau)$ such that

$$Mod_{\mathcal{L},\tau}(\varphi \wedge \psi) = Mod_{\mathcal{L},\tau}(\varphi) \cap Mod_{\mathcal{L},\tau}(\psi).$$

3. (Negation). For every signature τ and every sentence $\varphi \in \mathcal{L}(\tau)$, there is a sentence $\neg\varphi \in \mathcal{L}(\tau)$ such that

$$Mod_{\mathcal{L},\tau}(\neg\varphi) = Str(\tau) \setminus Mod_{\mathcal{L},\tau}(\varphi).$$

4. (Unary existential quantifier). For every signature τ and every sentence $\varphi \in \mathcal{L}(\tau \cup \{c\})$ there is a sentence $\exists c\varphi(c) \in \mathcal{L}(\tau)$ such that

$$Mod(\exists c\varphi(c)) = \{\mathcal{M} : \exists c \in \mathcal{M}\ (\mathcal{M}, c) \models \varphi\}.$$

Furthermore, a logic $\mathcal{L}$ is *small*, if for every signature τ, the collection $\mathcal{L}(\tau)$ of τ-sentences is a set.

Definition 2.8 (Regular logic)**.** A logic $\mathcal{L}$ is *regular* if it is small, it is closed under first-order operations and it has the atomic substitution property.

Further properties of regular logics

The *occurrence number* of a logic $\mathcal{L}$, if exists, is the least cardinal γ such that for every signature τ,

$$\mathcal{L}(\tau) = \bigcup_{\tau_0 \in [\tau]^{<\gamma}} \mathcal{L}(\tau_0).$$

If a logic $\mathcal{L}$ has an occurrence number, then for every signature τ and every sentence $\varphi \in \mathcal{L}(\tau)$, there is some minimal $\tau_\varphi \subseteq \tau$ of size at most γ such that $\varphi \in \mathcal{L}(\tau_\varphi)$.

The occurrence number of first-order logic is ω and the occurrence number of the logics $\mathcal{L}_{\kappa\lambda}$ is λ. See details for $\mathcal{L}_{\kappa\lambda}$ in [1] or [3].

The existence of an occurrence number allows to define the notion of elementary substructure. Indeed, say that $\mathcal{L}$ is a logic with occurrence number γ. Let τ be a signature and suppose that $\mathcal{M}$ and $\mathcal{N}$ are τ-structures. We say that $\mathcal{M}$ is an *$\mathcal{L}$-elementary submodel of* $\mathcal{N}$ and write $\mathcal{M} \preccurlyeq_{\mathcal{L}} \mathcal{N}$, if $\mathcal{M} \subseteq \mathcal{N}$ and for all tuples $\bar{a} \in \mathcal{M}^\gamma$,

$$(\mathcal{M}, \bar{a}) \equiv_{\mathcal{L}} (\mathcal{N}, \bar{a}).$$

For the logics $\mathcal{L}_{\kappa\lambda}$, we write $\mathcal{M} \preccurlyeq_{\kappa\lambda} \mathcal{N}$ as a shorthand for $\mathcal{M} \preccurlyeq_{\mathcal{L}_{\kappa\lambda}} \mathcal{N}$.

The *Löwenheim-Skolem number* of a logic $\mathcal{L}$, if exists, is the least λ such that every sentence that has a model, has one of size $\leqslant \lambda$.

The occurrence number and the Löwenheim-Skolem number of the first-order logic are both ω, the latter by the downwards Löwenheim-Skolem theorem. Similarly, in the logic $\mathcal{L}_{\kappa\lambda}$, in case κ large enough compared to λ, the proof generalizes to the infinite and gives that also the logic $\mathcal{L}_{\kappa\lambda}$ has a downwards Löwenheim-Skolem theorem. We state the theorem and refer the reader to [3] for a proof.

Theorem 2.9. *Assume that $\kappa > \beth_{\lambda^+}$. For every signature τ, for every τ-structure $\mathcal{M}$ and every subset $A \subseteq \mathcal{M}$, there is a τ-structure $\mathcal{N}$ of size at most $\kappa + |\tau| + |A|$ such that*

$$A \subseteq \mathcal{N} \preccurlyeq_{\kappa\lambda} \mathcal{M}.$$

In particular, the Löwenheim-Skolem number of $\mathcal{L}_{\kappa\lambda}$ is κ.

Remark 2.10. The Löwenheim-Skolem number reflects down: Let $\mathcal{L}$ and $\mathcal{L}'$ be logics. If $\mathcal{L} \leqslant \mathcal{L}'$ and $\mathcal{L}'$ has a Löwenheim-Skolem number, then so does $\mathcal{L}$, and the Löwenheim-Skolem number of $\mathcal{L}$ is at most that of $\mathcal{L}'$.

The logic $\mathcal{L}^1_\kappa$ which will be defined in the next section will be a sublogic of $\mathcal{L}_{\kappa\kappa}$, so in particular it will have a Löwenheim-Skolem number, which will be at most κ.

Remark 2.11. Every logic that has an occurrence number and a Löwenheim-Skolem number is equivalent to a small logic.

Proof. Suppose that λ is the maximum of the Löwenheim-Skolem number and the occurrence number of $\mathcal{L}$. For every signature τ and every sentence $\varphi \in \mathcal{L}(\tau)$, denote by τ_φ the minimal signature $\tau_\varphi \subseteq \tau$ such that $\varphi \in \mathcal{L}(\tau_\varphi)$. Define the logic $\mathcal{L}^*$ as follows:

1. For every signature τ and every sentence $\varphi \in \mathcal{L}(\tau)$, define the sentence φ^* to be the set of models

$$\varphi^* := Mod_{\mathcal{L},\tau_\varphi}(\varphi) \cap H(\lambda).$$

   and let $\mathcal{L}^*(\tau)$ consist of all the φ^* such that $\varphi \in \mathcal{L}(\tau)$.

2. For every signature τ and every τ-structure $\mathcal{M}$, define

$$\mathcal{M} \models_{\mathcal{L}^*} \varphi^* \quad :\Longleftrightarrow \quad \exists\, \mathcal{N} \in \varphi^* \;\; \mathcal{M} \equiv_{\mathcal{L}} \mathcal{N}.$$

Clearly every $\mathcal{L}^*(\tau)$ is a set. Furthermore, the logic $\mathcal{L}^*$ is equivalent to $\mathcal{L}$, because $Mod_{\mathcal{L}^*,\tau}(\varphi^*) = Mod_{\mathcal{L},\tau}(\varphi)$ for every signature τ and every sentence $\varphi \in \mathcal{L}(\tau)$. ☐

Projective classes

It is possible to fine tune measuring the expressive power of logics by taking into account those model classes that are *projective*, i.e. definable in an augmented signature. We briefly introduce projective classes, since they will be important when talking about interpolation.

For example, the class of infinite sets is projective in the first-order logic, since if the sentence

"f is an injection but not a surjection"

is satisfied in a structure, then its domain must be infinite, and conversely, every infinite set can be expanded into an $\{f\}$-structure that satisfies this sentence. Hence the class of infinite sets is definable in an augmented signature, i.e. it is projective.

Let τ be signature. A subset of an expansion of a τ-structure is τ-*closed* if it contains all the interpretations of the constant symbols of τ and it is closed under all the interpretations of the function symbols of τ. In this case we might also say that the subset is a *substructure*. For two signatures τ and σ, unary predicate symbol $P \in \sigma \setminus \tau$, and a $\tau \cup \sigma$-structure $\mathcal{M}$, in case the set $P^{\mathcal{M}}$ is τ-closed, write $(P^{\mathcal{M}}, \tau^{\mathcal{M}})$ for the structure obtained from $\mathcal{M}$ by restricting the domain to the set $P^{\mathcal{M}}$ and restricting the signature to τ.

Definition 2.12 (Projective classes)**.** Let $\mathcal{L}$ be a logic and let τ be a signature. Let $\mathcal{K}$ be a model class in the signature τ.

1. The class $\mathcal{K}$ is *projective (in $\mathcal{L}$)* if there is a signature σ and a $\tau \cup \sigma$-sentence φ such that

$$\mathcal{K} = \{(M, \tau^{\mathcal{M}}) : \mathcal{M} \models \varphi\}.$$

2. The class $\mathcal{K}$ is *relativized projective* if there is a signature σ with a unary predicate symbol $P \in \sigma \setminus \tau$ and a $\tau \cup \sigma$-sentence φ such that

$$\mathcal{K} = \{(P^{\mathcal{M}}, \tau^{\mathcal{M}}) : \mathcal{M} \models \varphi\}.$$

Clearly, every definable class is projective, and every projective class is relativized projective.

Notation. Let $\mathcal{L}$ be a logic and let τ and σ be mutually disjoint signatures. Let φ be an $\mathcal{L}(\tau \cup \sigma)$-sentence. Write

$$Mod_{\mathcal{L},\tau}(\exists \sigma \psi) := \{(M, \tau^{\mathcal{M}}) : \mathcal{M} \models \sigma\}.$$

Again, the subscript may be dropped and we may write simply $Mod(\exists \sigma \psi)$. Of course this $\exists \sigma \varphi$ might not be a sentence in the logic $\mathcal{L}$, but it can be thought either as a second order sentence or else it can be identified with the projective model class it defines. It can be called a *projective sentence.* In this notation a model class $\mathcal{K}$ is projective if it is of the form $Mod(\exists \sigma \varphi)$, for some signature σ disjoint from the signature of $\mathcal{K}$.

More on the infinitary logics $\mathcal{L}_{\kappa\lambda}$

Let κ and λ be infinite cardinals, such that $\lambda \leqslant \kappa$. The logic $\mathcal{L}_{\kappa\lambda}$ was defined in 2.3. In this logic, the *quantifier rank* of a formula, $qr(\varphi)$, is defined just as in the first-order logic, with the additional inductive clauses

$$qr\Big(\bigwedge_{i<\alpha} \varphi_i\Big) = \sup_{i<\alpha}(qr(\varphi_i))$$
$$qr(\exists \bar{x} \varphi) = qr(\varphi) + 1.$$

For two τ-structures $\mathcal{M}$ and $\mathcal{N}$, write $\mathcal{M} \equiv^{\beta}_{\kappa\lambda} \mathcal{N}$ if $\mathcal{M}$ and $\mathcal{N}$ agree on $\mathcal{L}_{\kappa\lambda}$-sentences of quantifier rank $\leqslant \beta$.

In order to display beautiful and useful characterizations of the relations $\equiv^{\beta}_{\kappa\lambda}$, we introduce Ehrenfeucht-Fraïssé games, partial isomorphisms and back-and-forth sets.

The Ehrenfeucht-Fraïssé games

The Ehrenfeuct-Fraïssé games are two-player games on two structures. The first player always picks subsets of the structures and the second player finds partial isomorphisms between the structures so that the sets chosen by the first player are contained in the field of the partial isomorphisms. This is the basic setup. We also introduce a variation, the *Ehrenfeucht-Fraïssé game with height* in which in addition, the first player descends along ordinals, which causes the game to be finite.

Let τ be a signature and let $\mathcal{M}$ and $\mathcal{N}$ be τ-structures. A *partial isomorphism* $f : \mathcal{M} \to \mathcal{N}$ is a partial map with domain included in $\mathcal{M}$ and range included in $\mathcal{N}$ such that the structures

$$(\mathcal{M}, (a)_{a \in dom(f)}) \quad \text{and} \quad (\mathcal{N}, (f(a))_{a \in dom(f)})$$

agree on atomic τ-sentences. If f and g are partial isomorphisms, then f *extends* g is $f \supseteq g$.

Let τ be a signature and let $\mathcal{M}$ and $\mathcal{N}$ be τ-structures. Suppose that $f : \mathcal{M} \to \mathcal{N}$ is a partial isomorphism. Let $a \in \mathcal{M} \cup \mathcal{N}$. Say that f *covers* the element a if $a \in dom(f) \cup ran(f)$. Similarly, a partial isomorphism covers a tuple or a set if it covers each element from the tuple or the set. Say that f can be *extended to cover* a if there is a partial isomorphism $g \supseteq f$ such that $a \in dom(g) \cup ran(g)$. Similarly for tuples and sets.

Definition 2.13 (Ehrenfeucht-Fraïssé game without height)**.** Let θ be a cardinal. Let τ be a signature and let $\mathcal{M}$ and $\mathcal{N}$ be τ-structures. The *Ehrenfeuch-Fraïssé game* $EF_\theta(\mathcal{M}, \mathcal{N})$ *of width* θ is played as follows.

> **Starting state**
> The player I plays the empty set $\emptyset$. The player II plays the empty map from $\mathcal{M}$ into $\mathcal{N}$. She loses if it fails to be a partial isomorphism, i.e. to preserve atomic sentences. Otherwise the game continues.
>
> **Further states**
> The player I plays some set $A \in [\mathcal{M}]^{\leqslant\theta} \cup [\mathcal{N}]^{\leqslant\theta}$. The player II plays some partial isomorphism $f : \mathcal{M} \to \mathcal{N}$ that covers A and extends all the partial isomorphisms that she played previously.

The player I wins if the player II fails to follow the rules at some round. Otherwise the player II wins.

Definition 2.14 (Ehrenfeucht-Fraïssé game with height)**.** Let θ be a cardinal and let β be an ordinal. Let τ be a signature. Let $\mathcal{M}$ and $\mathcal{N}$ be τ-structures. The *Ehrenfeuch-Fraïssé game* $EF_\theta(\mathcal{M}, \mathcal{N})$ *of height* β *and width* θ is played as follows.

> **Starting state**
> The player I plays the ordinal β and the empty set $\emptyset$. The player II plays the empty map from $\mathcal{M}$ into $\mathcal{N}$. She loses if it fails to be a partial isomorphism, i.e. to preserve atomic sentences. Otherwise the game continues.
>
> **Further states**
> The player I plays an ordinal α strictly below the one he played in the last state, and some set $A \in [\mathcal{M}]^{\leqslant\theta} \cup [\mathcal{N}]^{\leqslant\theta}$. The player II plays some partial isomorphism $f : \mathcal{M} \to \mathcal{N}$ that covers A and extends all the partial isomorphisms that she played previously.

The player who first breaks the rules loses.

Back and Forth

The second player having a winning strategy in the Ehrenfeucht-Fraïssé game can also be formulated in terms of partial isomorphisms. We do this.

Following the traditional definition, two structures $\mathcal{M}$ and $\mathcal{N}$ are *partially isomorphic*, $\mathcal{M} \cong_P \mathcal{N}$, if for any finite partial isomorphism $f : \mathcal{M} \to \mathcal{N}$ and for any element $a \in \mathcal{M} \cup \mathcal{N}$ there is a partial isomorphism $g \supseteq f$ such that $a \in dom(g) \cup ran(g)$. This definition extends to larger partial isomorphisms as well.

Definition 2.15. Let τ be a signature and let $\mathcal{M}$ and $\mathcal{N}$ be two τ-structures. Let θ be a cardinal.

1. The structures $\mathcal{M}$ and $\mathcal{N}$ are *θ-partially isomorphic*, $\mathcal{M} \cong_\theta \mathcal{N}$, if every partial isomorphism of size $< \theta$ can be extended to cover any subset of $\mathcal{M}$ or $\mathcal{N}$ of size $< \theta$.

2. A non-empty set J of partial isomorphisms f of size $< \theta$ from $\mathcal{M}$ to $\mathcal{N}$ is a *θ-back-and-forth set* if it satisfies the following *back-and-forth property*: *For any partial isomorphism $f \in J$ and for any subset $A \in [\mathcal{M}]^{\leqslant\theta} \cup [\mathcal{N}]^{\leqslant\theta}$, there is $g \in J$ that extends f and covers A.*
   If J is a θ-back-and-forth set for $\mathcal{M}$ and $\mathcal{N}$, write $J : \mathcal{M} \cong_\theta \mathcal{N}$.

Example 2.16. .

- If two structures $\mathcal{M}$ and $\mathcal{N}$ are countable and ω-partially isomorphic, then they are isomorphic. This is proved by first enumerating the domains of the structures and then recursively extending a partial isomorphism to cover each element.

- If two structures $\mathcal{M}$ and $\mathcal{N}$ are ω-partially isomorphic, then they are isomorphic in some forcing extension. It is enough to consider an ω-partial isomorphism set $J : \mathcal{M} \cong_\omega \mathcal{N}$ as a forcing poset, with the inverse inclusion as the ordering. The union of a generic filter is an isomorphism from $\mathcal{M}$ into $\mathcal{N}$.

Karp's theorems

Theorem 2.17 (Karp)**.** *Let θ be a cardinal and let $\lambda = \beth_{\theta^+}$. Assume that τ is a signature of size at most θ and $\mathcal{M}$ and $\mathcal{N}$ are τ-structures. The following are equivalent.*

1. $\mathcal{M} \equiv_{\lambda^+\theta^+} \mathcal{N}$.

2. $\mathcal{M} \cong_{\theta^+} \mathcal{N}$.

3. *The player II has a winning strategy in the Ehrenfeucht-Fraïssé game of width θ.*

4. *There is a partial isomorphism set J such that $J : \mathcal{M} \cong_{\theta^+} \mathcal{N}$.*

Proof. .

1.⇒2. Assume $\mathcal{M} \equiv_{\lambda^+\theta^+} \mathcal{N}$. Notice that there at most λ many pairwise non-equivalent formulas. If f is a partial isomorphism from $\mathcal{M}$ to $\mathcal{N}$ and $\bar{d}$ lists its domain, then for every $\bar{a} \in \mathcal{M}^{\leqslant\theta} \cup \mathcal{N}^{\leqslant\theta}$,

$$\mathcal{M} \models \exists \bar{x} \bigwedge tp(\bar{a}/\bar{d}) \iff \mathcal{N} \models \exists \bar{x} \bigwedge tp(\bar{a}/f(\bar{d})).$$

If $\bar{a} \in \mathcal{M}^{\leqslant\theta}$, then let $\bar{b} \in \mathcal{N}^{\leqslant\theta}$ be any witness for the existential quantifier, and the map $\bar{a} \mapsto \bar{b}$ extends f. If $\bar{a} \in \mathcal{N}^{\leqslant\theta}$, similarly any witness $\bar{b} \in \mathcal{M}^{\theta}$ gives an extension $\bar{b} \mapsto \bar{a}$ of f.

2.⇒3. Trivial.

3.⇒4. Assume that st is a winning strategy for the player II. It is a function

$$(g, A) \mapsto f,$$

where g is either $\emptyset$ or belongs to the range of st, $A \in [\mathcal{M}]^{\leqslant\theta} \cup [\mathcal{N}]^{\leqslant\theta}$ and f is a partial isomorphism that extends g and covers $\bar{a}$. Let J be the range of st. It is a θ^+-partial isomorhism set.

4.⇒1. Let $J : \mathcal{M} \cong_{\theta^+} \mathcal{N}$. We show that every map in J is elementary. We show by induction on the quantifier rank that every $f \in J$ preserves formulas of rank β.

Fix some f.

- $\beta = 0$: Trivial.
- β limit: By induction hypothesis.
- $\beta + 1$: Assume that f preserves formulas of rank β. It is enough to show that it preserves formulas of the form $\exists \bar{x} \psi(\bar{x}, \bar{y})$.

  Let $\bar{d}$ list the domain of f. Suppose that $\mathcal{M} \models \exists \bar{x} \psi(\bar{x}, \bar{d})$. Let $\bar{a} \in \mathcal{M}^{\leqslant\theta}$ witness $\mathcal{M} \models \varphi(\bar{a}, \bar{d})$. There is $g \in J$ such that it extends f and covers $\bar{a}$, and preserves formulas of rank β, by the induction hypothesis. Hence $\mathcal{N} \models \varphi(g(\bar{a}), f(\bar{d}))$, thus $\mathcal{N} \models \exists \bar{x} \varphi(\bar{x}, f(\bar{d}))$. It is symmetric to see that $\mathcal{N} \models \exists \bar{x} \varphi(\bar{x}, f(\bar{d}))$ implies $\mathcal{M} \models \exists \bar{x} \varphi(\bar{x}, \bar{d})$.

□

There is a weaker notion of partial isomorphism.

Definition 2.18. Let θ be a cardinal and let β be an ordinal. Two structures $\mathcal{M}$ and $\mathcal{N}$ are *θ-back-and-forth equivalent up to β*, $\mathcal{M} \cong_\theta^\beta \mathcal{N}$, if there is a sequence $(I_\alpha)_{\alpha \leqslant \beta}$ of non-empty sets of partial isomorphisms from $\mathcal{M}$ into $\mathcal{N}$, with the following back-and-forth property: *For any ordinal $\alpha < \beta$, for any partial isomorphism $f \in I_{\alpha+1}$ and for any set $A \in [\mathcal{M}]^{<\theta} \cup [\mathcal{N}]^{<\theta}$, there is $g \in I_\alpha$ that extends f and covers A.*

Theorem 2.19. *Let θ be a cardinal and let $\lambda = \beth_{\theta^+}$. Assume that τ is a signature of size at most θ and $\mathcal{M}$ and $\mathcal{N}$ are τ-structures. The following are equivalent.*

1. $\mathcal{M} \equiv^{\beta}_{\lambda^+\theta^+} \mathcal{N}$
2. $\mathcal{M} \cong^{\beta}_{\theta^+} \mathcal{N}$
3. *The player II has a winning strategy in the Ehrenfeucht-Fraïssé game of height β and width θ.*

Proof. The proof is similar to the theorem above.

1.⇒2. Assume that $\mathcal{M} \equiv^{\beta}_{\lambda^+\theta^+}$. The back-and-forth -sets are defined by

$$I_\alpha = \{f : f \text{ preserves formulas of quantifier rank } \alpha\}.$$

2.⇒3. Assume that $(I_\alpha)_{\alpha \leqslant \beta} : \mathcal{M} \cong^{\beta}_{\theta^+} \mathcal{N}$. The winning strategy for the player II is to choose a partial isomorphism from the set I_α whenever the player I is at height α.

3.⇒2. Assume that st is a strategy for the player II in the game $EF^{\beta}_{\theta}(\mathcal{M}, \mathcal{N})$. It is a function

$$((\alpha, g), \alpha', A) \mapsto (\alpha', f),$$

where α' and α are ordinals such that $\alpha < \alpha' \leqslant \beta$, $A \in \mathcal{M}^{\leqslant\theta} \cup \mathcal{N}^{\leqslant\theta}$, and g and f are partial isomorphisms, (α, g) belongs to the range of st and f extends g and covers A. The sets

$$I_\alpha := \{f : (\alpha, f) \in ran(st)\}$$

have the back-and-forth property up to β.

2.⇒1. Assume that $(I_\alpha)_{\alpha \leqslant \beta} : \mathcal{M} \cong^{\beta}_{\theta^+} \mathcal{N}$. As in the above proof, one shows by induction on α, that every $f \in I_\alpha$ preserves formulas of rank α. Then any $f \in I_\beta$ witnesses that $\mathcal{M} \equiv^{\beta}_{\lambda^+\theta^+} \mathcal{N}$.

□

Let $\mathcal{K}$ be a model class in some signature. Say that $\mathcal{K}$ is *closed under a game* if it is closed under the relation of the player II having a winning strategy in that game. In the first order logic, in a finite signature containing no function symbols, a model class is definable if and only if it is closed under the Ehrenfeucht-Fraïssé game of some finite height. The same holds in the infinitary setting.

Remark 2.20. Let κ be an infinite cardinal such that $\kappa = \beth_\kappa$. Let $\theta < \kappa$ be a cardinal and let $\lambda = \beth_{\theta^+}$. Let τ be a signature af size at most θ.

- For any ordinal β, there are at most λ many pairwise non-equivalent τ-sentences of quantifier rank at most β.
- For any ordinal β, every equivalence class of the relation $\equiv^{\beta}_{\lambda\theta}$ is definable in $\mathcal{L}_{\lambda\theta}(\tau)$, by the conjunction of the theory of one representative of that equivalence class, restricted to sentences of quantifier rank at most β.
- For any ordinal β, the relation $\equiv^{\beta}_{\lambda^+\theta^+}$ has at most λ equivalence classes.
- Every union of collection of equivalence classes of $\equiv^{\beta}_{\lambda^+\theta^+}$ is definable, by the disjunction of sentences that define each of the equivalence classes.

Let $\mathcal{K}$ be a model class in the signature τ. It is definable in the logic $\mathcal{L}_{\kappa\theta^+}$ if and only if there is an ordinal $\beta < \kappa$ such that $\mathcal{K}$ is closed under the relation $\equiv^{\beta}_{\lambda^+\theta^+}$. Equivalently, $\mathcal{K}$ is definable if and only if it is closed under the Ehrenfeucht-Fraïssé game of width θ and height β, for some ordinal $\beta < \beth_{\theta^+}$.

Example: A bizarre formulation of the logic $\mathcal{L}_{\kappa\theta}$

Let κ be an infinite cardinal such that $\kappa = \beth_\kappa$. We define an equivalent version of the logic $\mathcal{L}_{\kappa\theta}$ denoted by $\mathcal{L}^1_{\kappa\theta}$.

Define the logic $\mathcal{L}^1_{\kappa\theta}$ as follows.

- For a signature τ, an $\mathcal{L}^1_{\kappa\theta}(\tau)$-*sentence* is a model class in some signature $\tau_0 \in [\tau]^{<\kappa}$, which is closed under the Ehrenfeucht-Fraïssé game of width θ and height β, for some ordinal β.

  For a τ-sentence ψ, denote by τ_ψ the signature of the class ψ.
- The satisfaction relation is defined as

$$\mathcal{M} \models \psi \quad :\Longleftrightarrow \quad \mathcal{M} \restriction \tau_\psi \in \psi.$$

This logic $\mathcal{L}^1_{\kappa\theta}$ is equivalent to the logic $\mathcal{L}_{\kappa\theta}$, meaning that every model class definable in $\mathcal{L}_{\kappa\theta}$ is definable in $\mathcal{L}^1_{\kappa\theta}$ and vice versa.

This example is useless *per se*, but it serves as an introduction to the idea that a sentence of a logic is nothing else but purely the model class it defines. This will be the case for Shelah's logic - the sentences will be, by definition, model classes closed under a game.

3 Introducing the logic $\mathcal{L}^1_\kappa$

Shelah's game

Shelah's game is a variation of the Ehrenfeucht-Fraïssé game with height, with an additional feature that allows the second player delay her moves.

Recall that we say that a partial isomorphism f *covers* a set A if A is included in the field of f.

Definition 3.1 (The game $SG^\beta_\theta(\mathcal{M}, \mathcal{N})$)**.** Let τ be a signature and let $\mathcal{M}$ and $\mathcal{N}$ be τ-structures. Let β be an ordinal and let θ be a cardinal. The *Shelah's game $SG^\beta_\theta(\mathcal{M}, \mathcal{N})$ of height β and width θ* is played as follows.

> **Starting state**
> The player 1 plays the ordinal β and $\emptyset$. The player 2 plays the empty map from $\mathcal{M}$ into $\mathcal{N}$. She loses if it fails to be a partial isomorphism, i.e. to preserve atomic sentences. Otherwise the game continues.
>
> **Further states**
> The player 1 plays some ordinal β' strictly below the one he played in the previous state and some set $A \in [\mathcal{M}]^{\leqslant\theta} \cup [\mathcal{N}]^{\leqslant\theta}$, disjoint from his previously played ones.
>
> The player 2 assigns a finite height for each element in A. The elements that already have a height drop by one. In other words, the player 2 finds a partial function, a *height function*
>
> $$ht : M \cup N \to \omega$$
>
> such that if her previous height function was the function ht', then for all elements $b \in dom(ht')$,
>
> $$ht(b) = ht'(b) \dot{-} 1.$$
>
> In addition, she finds a partial isomorphism g which extends her previously chosen partial isomorphisms and which covers all the elements at height 0, i.e., $ht^{-1}\{0\} \subseteq dom(g) \cup ran(g)$.

The player who first cannot move loses.

In case the player 2 does not lose, the game looks like this.

1	$\beta, \emptyset$		β_1, A_1		$\dots$		$0, A_n$	
2		$\emptyset, \emptyset$		ht_1, g_1		$\dots$		ht_n, g_n

The g_i are partial isomorphisms such that $g_0 \subseteq g_1 \subseteq \cdots \subseteq g_n$. The ht_i are height functions. For any i,

$$A_i \subseteq dom(ht_i) \quad \text{and} \quad \text{for all } a \in dom(ht_i) \quad ht_{i+1}(a) = ht_i(a) \dot{-} 1.$$

The last partial isomorphism g_n covers all the elements that get thrown to height k or lower, at step $n - k$, for every $k \leqslant n$.

Remark 3.2. The game is finite and thus determined.

Remark 3.3. Any winning strategy for the second player in the Ehrenfeucht-Fraïssé game of height β and width θ extends to a winning strategy for the second player in Shelah's game of height β and width θ (by letting the height functions be constant zero functions), for any ordinal β and any cardinal θ.

Example 3.4. Let θ be a cardinal of uncountable cofinality. Let $\mathcal{M}$ and $\mathcal{N}$ be structures in some signature τ. Suppose that the player 2 has a winning strategy in the game $SG_\theta^{\omega+1}(\mathcal{M},\mathcal{N})$. If $|\mathcal{M}| < \theta$, then $|\mathcal{M}| = |\mathcal{N}|$.

Proof. Suppose to the contrary that $\mathcal{N}$ has size $\geqslant \theta$. As his first move, the player 1 should play a subset of $\mathcal{N}$ of size θ. Then the player 2 partitions this subset into countably many pieces with her height function. There is some n such that the nth piece has size θ. Next, if the player 1 plays n and keeps dropping by one integer at a time, the player 2 will lose before he touches the ground, because she would have to inject this piece of size θ into $\mathcal{M}$, which has size $< \theta$. Thus the player 2 cannot have a winning strategy. □

Example 3.5. Let $\mathcal{A} = (A, <_A)$ and $\mathcal{B} = (B, <_B)$ be any infinite dense linear orders without endpoints. The player 2 has a winning strategy in the game $SG_\omega^\beta(\mathcal{A},\mathcal{B})$ for any ordinal β.

Proof. We describe a winning strategy for the player 2: At each step the player 1 plays an ordinal and some countable set A. The player 2 first enumerates the set A, which gives the height function. At each step, there are only finitely many elements at height 0, so she can always map them, as both orderings are dense. This strategy is winning. □

The induced equivalence relations

The game SG_θ^β is reflexive and symmetric, i.e. the relation of the second player having a winning strategy is reflexive and symetric. It is not known whether the game is always transitive.

Definition 3.6. Let β be an ordinal and let θ be a cardinal. Let $\equiv_\theta^\beta$ be the transitive closure of the relation

$$\text{"}\textit{The player } 2 \textit{ has a winning strategy in } SG_\theta^\beta(\mathcal{M},\mathcal{N})\text{"}.$$

The relation $\equiv_\theta^\beta$ is the least equivalence relation on $Str(\tau)$ closed under the game SG_θ^β.

Remark 3.7. Assume that κ is an infinite cardinal such that $\kappa = \beth_\kappa$. Let $\beta < \kappa$ be an ordinal and let $\theta < \kappa$ be a cardinal.

1. For any ordinal $\beta' \geqslant \beta$ and cardinal $\theta' \geqslant \theta$, the relation $\equiv_{\theta'}^{\beta'}$ refines the relation $\equiv_\theta^\beta$.

2. Let $\lambda := \beth_{\theta^+}$. The relation $\equiv_{\lambda^+\theta^+}^\beta$ refines the relation $\equiv_\theta^\beta$. This is a consequence of the Remark 3.3 and Karp's theorem 2.19.

Deriving the logic

Definition 3.8 (The logic $\mathcal{L}^1_\kappa$)**.** Let θ^+ be a successor cardinal. For any signature τ, define:

1. An $\mathcal{L}^1_{\theta^+}(\tau)$-*sentence* is a class of τ_0-structures closed under some $\equiv^\beta_\theta$, where $\tau_0 \in [\tau]^{<\theta^+}$ is a subsignature and $\beta < \theta^+$ is an ordinal.

   For a τ-sentence φ, denote by τ_φ the signature such that φ consists of τ_φ-structures. Denote by β_φ and θ_φ the least ordinal β and cardinal θ such that φ is closed under the relation $\equiv^\beta_\theta$.

2. For a τ-structure $\mathcal{M}$ and a sentence $\varphi \in \mathcal{L}^1_{\theta^+}(\tau)$, let

$$\mathcal{M} \models \varphi \quad :\Longleftrightarrow \quad \mathcal{M} \restriction \tau_\varphi \in \varphi.$$

Let κ be a limit cardinal or ∞. Define

$$\mathcal{L}^1_\kappa = \bigcup_{\theta<\kappa} \mathcal{L}^1_{\theta^+}.$$

Remark 3.9. It is immediate to see that $\mathcal{L}^1_\kappa$ is a logic, for any cardinal κ.

From now onwards, let κ denote an uncountable cardinal such that $\kappa = \beth_\kappa$. We consider the logic $\mathcal{L}^1_\kappa$ solely for such cardinals.

Remark 3.10. A model class in a signature of size $< \kappa$ is definable in the logic $\mathcal{L}^1_\kappa$ if and only if there is an ordinal $\beta < \kappa$ and a cardinal $\theta < \kappa$ such that it is closed under the game SG^β_θ.

The logic $\mathcal{L}^1_\kappa$ is a regular logic

It is straightforward to see that the logic $\mathcal{L}^1_\kappa$ is a regular logic, as defined in 2.8. We need to verify that it is small, it is closed under first-order operations and it has the atomic substitution property.

Remark 3.11. The logic $\mathcal{L}^1_\kappa$ has occurrence number κ, by definition. We will observe in the Proposition 3.14 that it is a sublogic of $\mathcal{L}_{\kappa\kappa}$. These facts will give smallness: the bounded Löwenheim-Skolem number of $\mathcal{L}_{\kappa\kappa}$ reflects down to $\mathcal{L}^1_\kappa$, so by the Remark 2.11, $\mathcal{L}^1_\kappa$ will be (equivalent to) a small logic.

Proposition 3.12. *The logic $\mathcal{L}^1_\kappa$ is closed under atomic sentences, negation, conjunction and the existential quantifier.*

Proof. We check each case.

Atomic sentences: Assume that φ is an atomic sentence, in some signature τ. Without loss of generality τ is finite. We need to find an $\mathcal{L}^1_\kappa$-sentence φ', i.e. a model class closed under some $\equiv^\beta_\theta$, such that for any τ-structure $\mathcal{M}$,

$$\mathcal{M} \models \varphi \iff \mathcal{M} \restriction \tau_{\varphi'} \in \varphi'.$$

Let φ' to be the class of all τ-structures that satisfy φ (in particular, then $\tau_{\varphi'} = \tau$). This class is closed under the relation $\equiv^0_\theta$, for any cardinal θ, and works as wanted by definition.

Negation: Let τ be a signature. Let φ be an $\mathcal{L}^1_\kappa(\tau)$-sentence. Then $\neg\varphi$ can be taken as the complement of φ.

Conjunction: Let τ be a signature. Let φ and ψ be $\mathcal{L}^1_\kappa(\tau)$-sentences. We find an $\mathcal{L}^1_\kappa$-sentence $\varphi \wedge \psi$ such that

$$Mod_\tau(\varphi \wedge \psi) = Mod_\tau(\varphi) \cap Mod_\tau(\psi).$$

Let $\beta := max\{\beta_\varphi, \beta_\psi\}$ and $\theta := max\{\theta_\varphi, \theta_\psi\}$. Then φ and ψ are both closed under $\equiv^\beta_\theta$ and $\varphi \wedge \psi$ can be taken as the class of $\tau_\varphi \cup \tau_\psi$-structures $\mathcal{M}$ such that $\mathcal{M} \restriction \tau_\varphi \in \varphi$ and $\mathcal{M} \restriction \tau_\psi \in \psi$.

Existential quantifier: Let τ be a signature and let c be a constant symbol. Suppose that φ is an $\mathcal{L}^1_\kappa(\tau \cup \{c\})$-sentence. Let $\beta := \beta_\varphi + \omega + 1$ and let $\theta := \theta_\varphi$. Define

$$\exists c\varphi := \{\mathcal{M} \ : \ \exists c \ \ (\mathcal{M}, c) \in \varphi\}.$$

We need to show that the class $\exists c\varphi$ is an $\mathcal{L}^1_\kappa$-sentence. Let us show that it is closed under the relation $\equiv^\beta_\theta$.

First of all, if this class is empty, then it is an $\mathcal{L}^1_\kappa$-sentence. Otherwise, pick some structure $\mathcal{M} \in \exists c\varphi$ and suppose that the player 2 has a winning strategy in the game $SG^\beta_\theta(\mathcal{M}, \mathcal{N})$, for some τ-structure $\mathcal{N}$. We show that $\mathcal{N} \in \exists c\varphi$, by describing a play of the player 1. He may start by playing the ordinal $\beta_\varphi + \omega$ and the set $\{c\}$. Before he falls to height β_φ, the player 2 must have mapped the element c to some $d \in \mathcal{N}$, such that

$$(\mathcal{M}, c) \ \equiv^{\beta_\varphi}_\theta \ (\mathcal{N}, d).$$

As the sentence φ is closed under the relation $\equiv^{\beta_\varphi}_\theta$, we have $(\mathcal{N}, d) \in \varphi$, which gives that $\mathcal{N} \in \exists c\varphi$, as wanted.

□

Lemma 3.13. *The logic $\mathcal{L}^1_\kappa$ has the atomic substitution property.*

Proof. Suppose that φ is an $\mathcal{L}^1_\kappa$-sentence in some signature τ. By the occurrence number of $\mathcal{L}^1_\kappa$ being κ, we may assume that the signature τ has size $< \kappa$. Let P be an $(n+1)$-ary predicate, not from τ.

For a $\{P, \dots\}$-structure $\mathcal{A}$ and a tuple $\bar{a} \in \mathcal{A}^n$, we denote

$$P^{\mathcal{A}}_{\bar{a}} = \{b \in \mathcal{A} : P^{\mathcal{A}}(\bar{a}, b)\}.$$

Let τ^n be the signature obtained from τ by replacing each k-ary symbol s by an $(n+k)$-ary symbol s_n. For a $\tau^n \cup \{P, \dots\}$-structure $\mathcal{A}$ and a tuple $\bar{a} \in \mathcal{A}^n$, we denote by $(P_{\bar{a}}^{\mathcal{A}}, \tau_{\bar{a}}^{\mathcal{A}})$ the τ-structure with domain $P_{\bar{a}}^{\mathcal{A}}$ and interpretations

$$c \mapsto c_n^{\mathcal{A}}(\bar{a}), \quad f \mapsto f_n^{\mathcal{A}}(\bar{a}, \cdot) \quad \text{and} \quad R \mapsto \{\bar{b} : R_n^{\mathcal{A}}(\bar{a}, \bar{b})\},$$

for all constant symbols $c \in \tau$, function symbols $f \in \tau$ and relation symbols $R \in \tau$, and we say that $P_{\bar{a}}^{\mathcal{A}}$ is $\tau_{\bar{a}}$-*closed* if it is τ-closed under these interpretations.

We define a formula $\varphi_{\bar{x}}^P$ in the variable $\bar{x} = (x_i)_{i \in n}$, such that the following are equivalent for every $\tau^n \cup \{P, \dots\}$-structure $\mathcal{A}$ and for every tuple $\bar{a} \in \mathcal{A}^n$:

1. $\mathcal{A} \models \varphi_{\bar{a}}^P$.
2. If $P_{\bar{a}}^{\mathcal{A}}$ is $\tau_{\bar{a}}$-closed, then $(P_{\bar{a}}^{\mathcal{A}}, \tau_{\bar{a}}^{\mathcal{A}}) \models \varphi$.

Recall that the variables x_i, $i \in n$, are new constant symbols. Let $\varphi_{\bar{x}}^P$ be the class of structures in the signature $\tau^n \cup \bar{x} \cup \{P\}$ defined as follows:

$$\varphi_{\bar{x}}^P := \{\mathcal{M} : \textit{If } P_{\bar{x}}^{\mathcal{M}} \textit{ is } \tau_{\bar{x}}^{\mathcal{M}}\textit{-closed, then } (P_{\bar{x}}^{\mathcal{M}}, \tau_{\bar{x}}^{\mathcal{M}}) \models \varphi\}.$$

We have to see that $\varphi_{\bar{x}}^P$ is an $\mathcal{L}_\kappa^1(\tau^n \cup \{P\})$-formula, i.e. that it is an $\mathcal{L}_\kappa^1(\tau^n \cup \{P\} \cup \{x_i\}_{i \in n})$-sentence. This amounts to show that it is closed under some relation $\equiv_\theta^\beta$, for some β and θ.

Let $\beta := \beta_\varphi + \omega + 1$ and let $\theta := \theta_\varphi$. Assume that $\mathcal{M}$ and $\mathcal{N}$ are $\tau^n \cup \{P\} \cup \bar{x}$-structures, $\mathcal{M} \in \varphi_+^P$ and that the player 2 has a winning strategy in the game

$$SG_\theta^\beta(\mathcal{M}, \mathcal{N}).$$

Firstly, the predicate $P_{\bar{x}}^{\mathcal{M}}$ is $\tau_{\bar{x}}^{\mathcal{M}}$-closed if and only if the predicate $P_{\bar{x}}^{\mathcal{N}}$ is $\tau_{\bar{x}}^{\mathcal{N}}$-closed. Otherwise the player 2 loses before the player 1 reaches the height β_φ: the player 1 can play some elements that witness that $P_{\bar{x}}$ is not τ-closed, in any of the structures, and then pick a height to himself right above the elements and follow them until they reach the ground and he reaches the height β_φ.

In case $\mathcal{M}$ and $\mathcal{N}$ are both τ-closed, it suffices to show that the player 2 has a winning strategy in the game

$$SG_\theta^{\beta_\varphi}((P_{\bar{x}}^{\mathcal{M}}, \tau_{\bar{x}}^{\mathcal{M}}), (P_{\bar{x}}^{\mathcal{N}}, \tau_{\bar{x}}^{\mathcal{N}})).$$

But this is clear; if the player 1 plays inside $P_{\bar{x}^{\mathcal{M}}}^{\mathcal{M}} \cup P_{\bar{x}^{\mathcal{N}}}^{\mathcal{N}}$, then the winning strategy of 2 in the game on $\mathcal{M}$ and $\mathcal{N}$ gives her only moves that are legitimate moves in this game on $P_{\bar{x}}^{\mathcal{M}}$ and $P_{\bar{x}}^{\mathcal{N}}$ as well.

This shows that $\varphi_{\bar{x}}^P$ is indeed an $\mathcal{L}_\kappa^1$-formula. It is as wanted by definition. This finishes the proof that $\mathcal{L}_\kappa^1$ has the atomic substitution property. $\square$

The logic $\mathcal{L}^1_\kappa$ is between $\mathcal{L}_{\kappa\omega}$ and $\mathcal{L}_{\kappa\kappa}$

Recall that κ is assumed to be a cardinal such that $\kappa = \beth_\kappa$. We show that the logic $\mathcal{L}^1_\kappa$ is stronger than the logic $\mathcal{L}_{\kappa\omega}$ but does not exceed the logic $\mathcal{L}_{\kappa\kappa}$.

Proposition 3.14. *Let κ be a cardinal such that $\kappa = \beth_\kappa$. Then*

$$\mathcal{L}_{\kappa\omega} \leqslant \mathcal{L}^1_\kappa \leqslant \mathcal{L}_{\kappa\kappa}.$$

Proof. Let us check both inequalities separately.

- **To see $\mathcal{L}_{\kappa\omega} \leqslant \mathcal{L}^1_\kappa$:** It suffices to show that $\mathcal{L}_{\theta^+\omega} \leqslant \mathcal{L}^1_{\theta^+}$ for every cardinal $\theta < \kappa$. Fix thus some cardinal $\theta < \kappa$. It is enough to see that the logic $\mathcal{L}^1_{\theta^+}$ is closed under conjunctions of size $\leqslant \theta$.

  Let $\{\varphi_i,\}_{i<\alpha}$ be a set of $\mathcal{L}^1_{\theta^+}(\tau)$-sentences, for some ordinal $\alpha \leqslant \theta$. Let τ' be the union of the signatures τ_{φ_i}. It has size $\leqslant \theta$ by the regularity of θ^+. Then the class of τ'-structures

  $$\bigwedge_{i\in\alpha} \varphi_i := \{\mathcal{M} \in Str(\tau') \ : \ \forall i \quad \mathcal{M} \restriction \tau_{\varphi_i} \in \varphi_i\}$$

  is closed under the relation $\equiv^\beta_\theta$, where β is the supremum of the β_{φ_i}. This is because each φ_i is closed under the relation $\equiv^\beta_\theta$. Hence $\bigwedge_{i\in\alpha}\varphi_i$ is an $\mathcal{L}^1_\kappa$-sentence. Furthermore, it defines the class $\bigcap Mod(\varphi_i)$, as wanted.

- **To see $\mathcal{L}^1_\kappa \leqslant \mathcal{L}_{\kappa\kappa}$:** Let φ be an $\mathcal{L}^1_\kappa(\tau)$-sentence. Assume that it is closed under the relation $\equiv^\beta_\theta$, for some cardinal κ and ordinal $\beta < \kappa$. There is λ such that the relation $\equiv^\beta_{\lambda^+\theta^+}$ refines the relation $\equiv^\beta_{\theta^+}$. Every collection of $\equiv^\beta_{\lambda^+\theta^+}$-classes is definable in $\mathcal{L}_{\kappa\kappa}$. These follow from Remarks 2.20 and 3.7.(2). Thus there is an $\mathcal{L}_{\kappa\kappa}$-sentence equivalent to the sentence φ.

$\square$

Corollary 3.15. *The logic $\mathcal{L}^1_\kappa$ is a regular logic between $\mathcal{L}_{\kappa\omega}$ and $\mathcal{L}_{\kappa\kappa}$, with occurrence number κ and Löwenheim-Skolem number κ.*

Example of the expressive power of $\mathcal{L}^1_\kappa$: A Partition property

By the notation $X \to (\gamma)^{<\omega}$ we mean that for every coloring $f : [X]^{<\omega} \to 2$ there is a homogeneous set $H \subseteq X$ of size γ, which, in turn, means that the restricted function $f \restriction [H]^n$ is constant for every $n \in \omega$.

Example 3.16. The class of sets X such that $X \not\to (\omega_1)^{<\omega}$ is projectively definable in the logic $\mathcal{L}^1_\kappa$.

Proof. Let us work in the signature $\tau = \{f, 0, 1\}$. Let us show that the class

$$\{\mathcal{M} \in Str(\tau) : f^{\mathcal{M}} \text{ does not have uncountable homogeneous set}\}$$

is closed under the relation $\equiv_{\omega_1}^{\omega+1}$. Assume that $\mathcal{M}$ is a τ-structure such that $f^{\mathcal{M}}$ does not have an uncountable homogeneous set. Suppose that the player 2 has a winning strategy in the game $SG_{\omega_1}^{\omega+1}(\mathcal{M},\mathcal{N})$, for some $\mathcal{N}$. Suppose to the contrary that $f^{\mathcal{N}}$ has an uncountable homogeneous set. We describe a play by the player 1. The player 2 applies her winning strategy.

- The player 1 starts by playing the ordinal ω and an uncountable set $H \subseteq \mathcal{N}$ which is homogeneous for the function $f^{\mathcal{N}}$. The player 2 partitions the set H into $(H_i)_{i\in\omega}$, with her height function. So if ht is her height function, then the partition is given by letting $H_i = ht^{-1}\{i\}$, for every $i \in \omega$.
- The player 1 plays some natural number i such that the set H_i is uncountable.
- Before the player 1 touches the ground, the player 2 must map the set H_i into $\mathcal{M}$. She will fail because the image would produce a homogeneous set for $f^{\mathcal{M}}$. This is a contradiction.

□

Elementary submodels

The elementary submodel relation $\preccurlyeq_{\mathcal{L}}$ for an arbitrary logic $\mathcal{L}$, with occurrence number γ, is defined for two structures $\mathcal{M}$ and $\mathcal{N}$ as follows:

$$\mathcal{M} \preccurlyeq_{\mathcal{L}} \mathcal{N} \quad :\Longleftrightarrow \quad \mathcal{M} \subseteq \mathcal{N} \quad \text{and} \quad \forall \bar{a} \in \mathcal{M}^{<\gamma} \quad (\mathcal{M},\bar{a}) \equiv_{\mathcal{L}} (\mathcal{N},\bar{a}).$$

A logic $\mathcal{L}$ has the *Tarski Union Property* if countable $\preccurlyeq_{\mathcal{L}}$-chains are closed under union, i.e. if $(\mathcal{M}_n)_n$ is a $\preccurlyeq_{\mathcal{L}}$-chain, then every $\mathcal{M}_n$ is an $\mathcal{L}$-elementary submodel of the union of the chain.

The logic $\mathcal{L}_\kappa^1$ does not have the Tarski Union property. However, we will define a weaker relation $\preccurlyeq_\varphi$, an elementary submodel relation relative to a formula φ, and we will define a weakened version of the Tarski Union property, by only requiring that each model in a $\preccurlyeq_\varphi$-chain agrees on the formula φ with the union of the chain. The logic $\mathcal{L}_\kappa^1$ will satisfy this weakened version of the Tarski Union property.

The relations $\preccurlyeq_\psi$

Let τ be a signature and let $\mathcal{M}$ and $\mathcal{N}$ be τ-structures. For a τ-sentence φ, write $\mathcal{M} \equiv_\psi \mathcal{N}$ if the structures $\mathcal{M}$ and $\mathcal{N}$ agree on the sentence ψ.

The next definition is due to B. Veličković and J. Väänänen, [15].

Definition 3.17. Let $\mathcal{L}$ be a logic, let τ be a signature and let ψ some $\mathcal{L}(\tau)$-formula. A relation $\preccurlyeq_\psi$ between τ-structures is a *ψ-submodel relation* if it has the following properties for all τ-structures $\mathcal{M}, \mathcal{N}$ and $\mathcal{P}$.

1. If $\mathcal{M} \preccurlyeq_\psi \mathcal{N}$, then $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{M} \equiv_\psi \mathcal{N}$.
2. If $\mathcal{M} \preccurlyeq_\psi \mathcal{N} \preccurlyeq_\psi \mathcal{P}$, then $\mathcal{M} \preccurlyeq_\psi \mathcal{P}$.
3. (Tarski-Vaught Property). If $\mathcal{M}, \mathcal{N} \preccurlyeq_\psi \mathcal{P}$ and $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{M} \preccurlyeq_\psi \mathcal{N}$.
4. (Isomorphic correction). Suppose that $f : \mathcal{M} \cong \mathcal{P}$. Then for any $\mathcal{N} \subseteq \mathcal{M}$,
$$\mathcal{N} \preccurlyeq_\psi \mathcal{M} \iff f[\mathcal{N}] \preccurlyeq_\psi \mathcal{P}.$$
5. (Extending first-order $\preccurlyeq_{\omega\omega}$). If $\mathcal{M} \preccurlyeq_\psi \mathcal{N}$, then $\mathcal{M} \preccurlyeq_{\omega\omega} \mathcal{N}$.

Assume that $\mathcal{L}$ is a logic which has a ψ-submodel relation $\preccurlyeq_\psi$ for each formula ψ in each signature τ. We define two additional properties that the relation might satisfy.

- (Union Property). Suppose that
$$\mathcal{M}_0 \preccurlyeq_\psi \mathcal{M}_1 \preccurlyeq_\psi \ldots \preccurlyeq_\psi \mathcal{M}_i \preccurlyeq_\psi \ldots$$
is a countable chain of τ-structures and let $\mathcal{M} := \bigcup \mathcal{M}_i$. For every i, the structures $\mathcal{M}_i$ and $\mathcal{M}$ agree on ψ.
- (Löwenheim-Skolem number). There is λ such that for every structure $\mathcal{M}$ in the signature of ψ and for every subset $A \subseteq \mathcal{M}$ of size λ, there is a structure $\mathcal{N}$ of size at most λ, such that
$$A \subseteq \mathcal{N} \preccurlyeq_\psi \mathcal{M}.$$
If exists, this λ is called the *Löwenheim-Skolem number of ψ.*

We show that the logic $\mathcal{L}^1_\kappa$ has relations $\preccurlyeq_\psi$ for each formula ψ, which each satisfy the Union property and have a Löwenheim-Skolem number strictly below κ.
Let τ be a signature. Let $\bar{x} = (x_i)_{i \in I}$ be a tuple of variables. A τ-formula $\varphi(\bar{x})$ in the variable $\bar{x}$ is a $\tau \cup \{x_i\}_{i \in I}$-sentence. In the logic $\mathcal{L}^1_\kappa$, as its occurrence number is κ, any formula has less than κ many free variables.

Definition 3.18 (The relations $\preccurlyeq_\psi$ for $\mathcal{L}^1_\kappa$)**.** Assume that τ is a signature and assume that ψ is an $\mathcal{L}^1_\kappa(\tau)$-formula. Let $\theta := \theta_\psi$ and let $\lambda := \beth_{\theta^+}$. Define $\preccurlyeq_\psi$ to be the elementary submodel relation $\preccurlyeq_{\lambda^+\theta^+}$ (from the logic $\mathcal{L}_{\lambda^+\theta^+}$).

Lemma 3.19 (The Union Lemma)**.** *Assume that ψ is an $\mathcal{L}^1_\kappa$-sentence and $(\mathcal{M}_i)_{i\in\omega}$ is a countable $\preccurlyeq_\psi$-chain. Let $\mathcal{M}$ be its union. Then for every $i \in \omega$,*
$$\mathcal{M}_i \equiv_\psi \mathcal{M}.$$

Proof. Assume that $(\mathcal{M}_i)_{i\in\omega}$ is an $\preccurlyeq_\psi$-chain and let $\mathcal{M} := \bigcup_{i\in\omega}\mathcal{M}_i$. Let $\beta := \beta_\psi$, $\theta := \theta_\psi$ and $\lambda := \beth_{\theta^+}$. Fix a natural number i. We show that the player 2 has a winning strategy in the game

$$SG^\beta_\theta(\mathcal{M}_i, \mathcal{M}).$$

Before starting the proof, recall that if two structures are $\mathcal{L}_{\lambda^+\theta^+}$-equivalent, then they are θ^+-partially isomorphic, by Karp's theorem 2.17. This gives that for any structures $\mathcal{N}$ and $\mathcal{P}$, if $\mathcal{N} \preccurlyeq_\psi \mathcal{P}$, then for all tuples $\bar{a} \in \mathcal{N}^{\leqslant\theta}$,

$$(\mathcal{N}, \bar{a}) \cong_{\theta^+} (\mathcal{P}, \bar{a}).$$

We describe a winning strategy for the player 2 in the game on $\mathcal{M}_i$ and $\mathcal{M}$.

- She does not lose the starting state.
- Assume that the player 1 does not touch the ground yet and he plays some $A \in \mathcal{M}_i^{\leqslant\theta} \cup \mathcal{M}^{\leqslant\theta}$. We may suppose that the player 1 plays always from the model $\mathcal{M}^{\leqslant\theta}$, because $\mathcal{M}_i \subseteq \mathcal{M}$. The player 2 then should partition the set A according to the chain, which means that she assigns the height j to elements in A that are in the set $\mathcal{M}_j - \bigcup_{k<j}\mathcal{M}_k$, for every j.

  At each round, the elements at height 0 are contained in finitely many models. There is thus at each round some k above the fixed natural number i, such that the model $\mathcal{M}_k$ contains all the elements from height 0.

  The player 2 should play as follows. Denote the set of elements currently at height 0 by B. The size of B is at most θ. Denote by g the partial isomorphism constructed so far by the player 2. There is k such that the range of g is contained in $\mathcal{M}_k$. The structures $\mathcal{M}_i$ and $\mathcal{M}_k$ are θ^+-partially isomorphic, so there is some partial isomorphism g' from $\mathcal{M}_i$ to $\mathcal{M}_k$ that extends g and covers B. The player 2 plays g' and does not lose.

□

Corollary 3.20. *For every $\mathcal{L}^1_\kappa$-sentence ψ, the relation $\preccurlyeq_\psi$ is a ψ-submodel relation that has the Union Property and Löwenheim-Skolem number strictly below κ.*

4 The logic $\mathcal{L}^1_\kappa$ from above

We present a proof of a Lindström theorem for $\mathcal{L}^1_\kappa$. The proof is due to S. Shelah, [11]. The alternative formulation of the theorem in terms of the ψ-submodel relations is due to B. Veličković and J. Väänänen, [15].

A *Lindström theorem* for an abstract logic $\mathcal{L}$ has the following form: *A logic $\mathcal{L}$ is maximal with respect to properties $P_1, \ldots P_k$.* The original proof by Lindström, [9], is for the first-order logic, with respect to the properties of being countably compact and having Löwenheim-Skolem number ω.

Among abstract logics, the proof method employed by Lindström generalizes to the logic $\mathcal{L}_{\kappa\omega}$ and to Shelah's logic $\mathcal{L}^1_\kappa$. The logic $\mathcal{L}_{\kappa\omega}$ is maximal with respect to having well-ordering number κ and the so-called *Karp Property*, which states that partially isomorphic models are elementarily equivalent in the logic in question.

Let us sketch a generic proof of a Lindström theorem, for an arbitrary logic, following the proof method employed by Lindström. Suppose thus that we want to show that a logic $\mathcal{L}$ is maximal with respect to some properties. Let us assume that this logic is regular and that it has an occurrence number.

The proof is by contradiction. One supposes to the contrary that there is some strictly stronger logic $\mathcal{L}' > \mathcal{L}$ which satisfies the properties in question. The fact that the logic $\mathcal{L}'$ is strictly stronger means that we may fix some $\mathcal{L}'$-sentence ψ which defines a model class not definable in the logic $\mathcal{L}$.

For the proof to go through, the logic $\mathcal{L}$ must have the following properties.

1. *Existence of a hierarchy $(\equiv_\beta)_{\beta \in I}$ that approximates $\equiv_\mathcal{L}$.*

   This hierarchy must approximate $\equiv_\mathcal{L}$ in the sense that being closed under some $\equiv_\beta$ must be a necessary and sufficient condition for a model class to be definable (at least for model classes in a signature smaller than the occurrence number).

   In the case of the first-order logic, the hierarchy is given by the relations $\equiv^k_{\omega\omega}$, equivalence up to quantifier rank k, and similarly in the logic $\mathcal{L}_{\kappa\omega}$ the hierarchy is given by $\equiv^\beta_{\kappa\omega}$, equivalence up to quantifier rank β. In the case of Shelah's logic, the hierarchy is given by the relations $\equiv^\beta_{|\beta|}$, induced by the game specially designed so that Lindström's proof goes through.

   The existence of such a hierarchy allows us, from the counterassumption that there is an $\mathcal{L}'$-sentence ψ not equivalent to any sentence in $\mathcal{L}$, infer that for every β, we may pick models $\mathcal{M}_\beta$ and $\mathcal{N}_\beta$ such that

   $$\mathcal{M}_\beta \equiv_\beta \mathcal{N}_\beta \quad \wedge \quad \mathcal{M}_\beta \models \psi \quad \wedge \quad \mathcal{N}_\beta \models \neg\psi,$$

   for otherwise the class $Mod(\psi)$ would be closed under some $\equiv_\beta$, thus definable.

2. *The logic $\mathcal{L}$ must be able to talk about the relations $\equiv_\beta$.*

   This means that the logic $\mathcal{L}$ must contain a sentence that expresses

   $$\forall \beta \in I \quad M_\beta \equiv_\beta N_\beta \quad \wedge \quad M_\beta \models \psi \quad \wedge \quad N_\beta \models \neg\psi,$$

   where M, N and I are predicate symbols and β is a variable. The notation "$M_\beta \equiv_\beta N_\beta$" means that in a structure $\mathcal{A}$, the sets $\{x : M^{\mathcal{A}}(\beta, x)\}$ and $\{x : N^{\mathcal{A}}(\beta, x)\}$ are $\equiv_\beta$-equivalent, in the sense which will be made explicit. Similarly, the $M_\beta \models \psi$ means that for a structure $\mathcal{A}$, the substructure with domain $\{x : M^{\mathcal{A}}(\beta, x)\}$ satisfies the sentence ψ. Notice that this is always definable by the atomic substitution property of a regular logic.

   In the cases of the first-order logic and the logic $\mathcal{L}_{\kappa\omega}$, the equivalence up to a certain quantifier rank is possible to code via the back-and-forth sets, thanks to Karp's theorem 2.19. The case of the logic $\mathcal{L}^1_\kappa$ is similar, even if the possible non-transitivity of Shelah's game will slightly complicate things.

3. *The properties in terms of which the logic $\mathcal{L}$ is characterized must include some form of a compactness property.*

   This requirement will guarantee that there will be some "non-standard" model with some "non-standard" β, such that the one instance $M_\beta \equiv_\beta N_\beta$ can be turned into an $\mathcal{L}'$-equivalence. This will give a contradiction, since it was assumed that M_β and N_β disagree on ψ.

   In the case of the first-order logic, there would be a countable model with a non-standard integer k, which will give that the predicates M_k and N_k are partially isomorphic. Thanks to the ill-foundedness of k, a partial isomorphism set of length k can be turned "upside down" to produce a real partial isomorphism. Then, by the fact that they are countable, they are isomorphic, and hence cannot disagree on ψ. In the case of the logic $\mathcal{L}_{\kappa\omega}$, with an ill-founded β, the predicates M_β and N_β would similarly be partially isomorphic, and by the Karp property, they would agree on ψ. In the case of the logic $\mathcal{L}^1_\kappa$, there will be a model with an ill-founded ordinal β and with a specific property that allows one turn the predicates isomorphic.

   If, in this third step, the predicates can be turned isomorphic, then the logic $\mathcal{L}$ will have interpolation, by the same proof.

The Compactness property of $\mathcal{L}^1_\kappa$

We isolate the compactness property in terms of which we will be then able to characterize the logic $\mathcal{L}^1_\kappa$. This property is a technical strengthening of the property of having well-ordering number κ. We will call it the *strong* well-ordering number κ. Shelah calls it in [11] the *strong undefinability of well-order at* κ, or $SUDWO_\kappa$, for short. Roughly, this property states that every sentence which has a "standard" model of the form $(H(\lambda), \in, \kappa, \dots)$, also has a model which is highly "non-standard"; ill-founded, and in addition, decomposes into countably many blocks of size $< \kappa$, in the sense which is made explicit in the statement of the next lemma.

Lemma 4.1 (The Sudwo Lemma)**.** *Assume that $\mathcal{L}$ is a regular logic that has ψ-submodel relations $\preccurlyeq_\psi$ with the Union Property and the Löwenheim-Skolem number below κ, for each formula ψ. Let τ be a signature and assume that ψ is an $\mathcal{L}(\tau)$-sentence that has a model of the form*

$$\mathcal{A} = (H(\lambda), \in, \kappa, \dots),$$

for some $\lambda \geqslant \kappa$. Here κ is a unary predicate. Then the sentence ψ has also a model

$$\mathcal{B} = (B, \in^{\mathcal{B}}, \kappa^{\mathcal{B}}, \dots)$$

with some infinite $\beta \in \kappa^{\mathcal{B}}$ and some $\{B_i\}_{i\in\omega} \subseteq \mathcal{B}$ such that

1. *$(\beta, \in^{\mathcal{B}})$ is ill-founded.*
2. *$\mathcal{B} \models \forall x \bigvee_{i\in\omega} x \in B_i$.*
3. *$\mathcal{B} \models \bigwedge_{i\in\omega} |B_i| \leqslant |\beta|$.*

Proof. Assume that ψ is some $\mathcal{L}(\tau)$-sentence which has a model $\mathcal{A}$ of the form

$$\mathcal{A} = (H(\lambda), \kappa, \in, \dots).$$

Let θ be the Löwenheim-Skolem number of ψ. We recursively build a chain of models of size at most θ. Their union will give the non-standard model $\mathcal{B}$ of ψ.

- **At step** 0**:** For every $\beta \in \kappa \setminus \theta$, choose some model $B^0_\beta \preccurlyeq_\psi \mathcal{A}$ containing β and $|\beta|$, of size $\leqslant \theta$. There are at most 2^θ pairwise non-isomorphic ones, so we can fix an unbounded set $X_0 \subseteq (2^\theta)^+$ such that for all $\beta, \beta' \in X_0$, there is an isomorphism
$$B^0_\beta \cong B^0_{\beta'},$$
that maps $\beta \mapsto \beta'$.

  Let (P_0, c_0) be some representative of this isomorphism type. For every $\beta \in X_0$, there is an isomorphism
$$B^0_\beta \cong P_0$$
which maps $\beta \mapsto c_0$.

- **At step** 1**:** The set X_0 is unbounded in $(2^\theta)^+$. For every infinite $\beta \in (2^\theta)^+$, choose some $x_0(\beta) \in X_0$ above β, and choose some $B^1_\beta \preccurlyeq_\psi \mathcal{A}$ of size $\leqslant \theta$, such that
$$B^0_{x_0(\beta)} \cup \{\beta\} \cup \{B^0_{x_0(\beta)}\} \subseteq B^1_\beta.$$

Notice that the Tarski-Vaught property implies that $B^0_{x_0(\beta)} \preccurlyeq_\psi B^1_\beta$.

There are at most 2^θ pairwise non-isomorphic structures of size θ, so we can fix an unbounded set $X_1 \subseteq (2^\theta)^+$ such that for all $\beta, \beta' \in X_1$, there is an isomorphism

$$B^1_\beta \cong B^1_{\beta'},$$

which maps

$$\beta \mapsto \beta' \quad \text{and} \quad B^0_{x_0(\beta)} \mapsto B^0_{x_0(\beta')}.$$

Let (P_1, c_1, D_1) be a representative of the isomorphism type. Both c_1 and D_1 are treated as elements or constants, not predicates. For every $\beta \in X_1$, there is an isomorphism

$$B^1_\beta \cong P_1$$

which maps $\beta \mapsto c_1$ and $B^0_{x_0(\beta)} \mapsto D_1$.

Furthermore, the isomorphisms induce an embedding $i_0 : P_0 \hookrightarrow P_1$ with the following property:

$$P_1 \models c_1 \in i_0(c_0) \quad \text{and} \quad P_1 \models i_0[P_0] \subseteq D_1.$$

For any $\beta \in X_1$, we have the diagram:

$$\begin{array}{ccc} B^0_{x_0(\beta)} & \subseteq & B^1_\beta \\ \downarrow \cong & & \downarrow \cong \\ P_0 & \overset{i_0}{\dashrightarrow} & P_1 \end{array}$$

- **At step $n+1$:** Assume that the set X_n and the structure (P_n, c_n, D_n) have been defined. Again, for every infinite $\beta \in (2^\theta)^+$, choose some $x_n(\beta) \in X_n$ above β, together with a stucture $B^{n+1}_\beta \preccurlyeq_\psi \mathcal{A}$ of size at most θ, such that

$$B^n_{x_n(\beta)} \cup \{\beta\} \cup \{B^n_{x_n(\beta)}\} \subseteq B^{n+1}_\beta.$$

Again, the Tarski-Vaught property implies that $B^n_{x_n(\beta)} \preccurlyeq_\psi B^{n+1}_\beta$.

There are at most 2^θ pairwise non-isomorphic such models, therefore we can fix an unbounded set $X_{n+1} \subseteq (2^\theta)^+$ such that for all $\beta, \beta' \in X_{n+1}$, there is an isomorphism

$$B^{n+1}_\beta \cong B^{n+1}_{\beta'}$$

which maps

$$\beta \mapsto \beta' \quad \text{and} \quad B^n_{x_n(\beta)} \mapsto B^n_{x_n(\beta')}.$$

Choose a representative $(P_{n+1}, c_{n+1}, D_{n+1})$ of this isomorphism type. Both c_{n+1} and D_{n+1} are elements of P_{n+1}. The isomorphisms induce an embedding $i_n : P_n \hookrightarrow P_{n+1}$ such that the model P_{n+1} satisfies

$$P_{n+1} \models c_{n+1} \in i_n(c_n) \quad \text{and} \quad P_{n+1} \models i_n[P_n] \subseteq D_{n+1}.$$

Now assume that the structures (P_n, c_n, D_n) and the embeddings $i_n : P_n \to P_{n+1}$ have been constructed for each n. Up to renaming, each i_n is an inclusion. Let

$$\mathcal{B} = \bigcup_n P_n.$$

Now $\mathcal{B}$ must be a model of ψ, because each B^n_β is a model of ψ, and by the isomorphic correction each P_n is a model of ψ, and therefore, by the Union property, also $\mathcal{B}$ is a model of ψ. We need to show that there is some $\beta \in \kappa^{\mathcal{B}}$ together with some $\{B_i\}_{i\in\omega} \subseteq \mathcal{B}$ such that

1. $(\beta, \in^{\mathcal{B}})$ is ill-founded.
2. $\mathcal{B} \models \forall x \bigvee_{i\in\omega} x \in B_i$.
3. $\mathcal{B} \models \bigwedge_{i\in\omega} |B_i| \leqslant |\beta|$.

Let $\beta := c_0$. Clearly it is ill–founded. For each i, let $B_i := D_i$. Clearly

$$\mathcal{B} \models \forall x \bigvee_{i\in\omega} x \in B_i.$$

There remains to see that $\mathcal{B} \models |B_i| \leqslant |\beta|$ for each $i \in \omega$. However, at the first step of the construction, we chose the set X_0 above θ, i.e. such that for every $\beta' \in X_0$,

$$\mathcal{A} \models |B_i| \leqslant |\beta'|, \quad \text{for each } i \in \omega.$$

But then also

$$\mathcal{B} \models |B_i| \leqslant |\beta|, \quad \text{for each } i \in \omega,$$

because $\mathcal{B}$ is first-order equivalent with $\mathcal{A}$.

□

Undefinability of well-order

The Sudwo-lemma has as a corollary that the well-ordering number of $\mathcal{L}^1_\kappa$ is at most κ. To see that the well-ordering number of the logic $\mathcal{L}^1_\kappa$ is exactly κ, recall that the logic $\mathcal{L}_{\kappa\omega}$ has well-ordering number κ, and that $\mathcal{L}_{\kappa\omega} \leqslant \mathcal{L}^1_\kappa$. This is enough to see that the well-ordering number of $\mathcal{L}^1_\kappa$ is at least κ.

Let us isolate the property of The Sudwo Lemma 4.1 above. The *strong well-ordering number* of a logic $\mathcal{L}$ is, if exists, the least cardinal γ such that the following property holds: *Every sentence which has a "standard model" of the form*

$$\mathcal{A} = (H(\lambda), \in, \gamma, \dots),$$

where γ is a predicate, also has a "non-standard" model $\mathcal{B} = (B, \in^{\mathcal{B}}, \gamma^{\mathcal{B}}, \dots)$ with an infinite $\beta \in \gamma^{\mathcal{B}}$ and some $\{B_i\}_{i\in\omega} \subseteq \mathcal{B}$ such that

1. *$(\beta, \in^{\mathcal{B}})$ is ill-founded.*

2. $\mathcal{B} \models \forall x \bigvee_{i\in\omega} x \in B_i$.

3. $\mathcal{B} \models \bigwedge_{i\in\omega} |B_i| \leqslant |\beta|$.

Observation 4.2. *Every logic which has the strong well-ordering number at most κ has the well-ordering number at most κ. Every logic which has the well-ordering number at least κ also has the strong well-ordering number at least κ.*

Corollary 4.3. *The strong well-ordering number of the logic $\mathcal{L}^1_\kappa$ is κ.*

Corollary 4.4. *The well-ordering numbers of the logics $\mathcal{L}^1_\kappa$ and $\mathcal{L}_{\kappa\omega}$ are κ.*

Lindström Theorem for $\mathcal{L}^1_\kappa$

Assume that τ is a signature and $\mathcal{M}$ and $\mathcal{N}$ are τ-structures. Let β be an ordinal and let θ be a cardinal. Consider the game

$$SG^\beta_\theta(\mathcal{M}, \mathcal{N}).$$

- A triple $t = (\alpha_t, ht_t, g_t)$ is a *state in the game* $SG^\beta_\theta(\mathcal{M}, \mathcal{N})$ if $\alpha_t \in \beta$ is an ordinal, ht_t is a partial height function from $\mathcal{M} \cup \mathcal{N}$ into ω and g_t is a partial isomorphism from $\mathcal{M}$ into $\mathcal{N}$.
- A state s *extends* another state t if there is a play in which none of the players break the rules and the state s comes after the state t in the game. In other words, s *extends* t if
  1. $\alpha_s \in \alpha_t$.
  2. for all $a \in dom(ht_t)$, $ht_s(a) < max\{1, ht_t(a)\}$.
  3. $g_s \supseteq g_t$.

Observation 4.5. *Assume that τ is a signature and $\mathcal{M}$ and $\mathcal{N}$ are τ-structures. The following are equivalent.*

1. *The player 2 has a winning strategy in the game $SG^\beta_\theta(\mathcal{M}, \mathcal{N})$.*
2. *There is a set* st *that codes a winning strategy for 2, in the following sense:*
   - *Every $t \in \mathrm{st}$ is a state in the game $SG^\beta_\theta(\mathcal{M}, \mathcal{N})$.*
   - *The starting state $(\beta, \emptyset, \emptyset)$ is in* st.
   - *For every $t \in \mathrm{st}$, for every $\alpha < \alpha_t$ and for every $A \in [\mathcal{M}]^{\leqslant\theta} \cup [\mathcal{N}]^{\leqslant\theta}$, there is $s \in \mathrm{st}$ that extends t and such that $\alpha_s = \alpha$ and $A \subseteq ht_s$.*

The First Characterization Theorem

We prove the following proposition, which will give the Lindström Theorem for $\mathcal{L}^1_\kappa$ as a corollary. It will also imply that the logic $\mathcal{L}^1_\kappa$ has the interpolation property.

Let τ be a signature. Recall that a model class is *relativized projective* if it is of the form

$$\{(P^{\mathcal{N}}, \tau^{\mathcal{N}}) : \mathcal{N} \models \varphi\},$$

where φ is a sentence in a larger signature augmenting τ, and P is a unary predicate in $\sigma \setminus \tau$.

Proposition 4.6. *Assume that $\mathcal{L}$ is a regular logic above the logic $\mathcal{L}_{\kappa\omega}$ with the occurrence number at most κ and with the strong well-ordering number at most κ. Then any two disjoint classes which are relativized projective in the logic $\mathcal{L}$ are separable by a class which is definable in the logic $\mathcal{L}^1_\kappa$.*

Proof. Let $\mathcal{L} \geqslant \mathcal{L}_{\kappa\omega}$ be a regular logic with the strong well-ordering number at most κ. Let τ be a signature and let $\mathcal{K}$ and $\mathcal{K}'$ be some disjoint relativized projective classes in the signature τ. Say that ψ and ψ' are $\mathcal{L}$-sentences in some signature $\sigma \supseteq \tau$ such that

$$\begin{aligned}\mathcal{K} &= \{(P^{\mathcal{N}}, \tau^{\mathcal{N}}) : \mathcal{N} \models \psi\}\\ \mathcal{K}' &= \{(P^{\mathcal{N}}, \tau^{\mathcal{N}}) : \mathcal{N} \models \psi'\}.\end{aligned}$$

Suppose to the contrary that $\mathcal{K}$ and $\mathcal{K}'$ are not separable by any $\mathcal{L}^1_\kappa(\tau)$-sentence. This means that for each $\beta \in \kappa$, there are σ-structures $\mathcal{M}_{\beta,i}$, $i \leqslant k_\beta$, for some integer k_β, such that

- $\mathcal{M}_{\beta,0} \models \psi$ and $\mathcal{M}_{\beta,k_\beta} \models \psi'$.
- For each $\beta \in \kappa$ and $i < k_\beta$, the player 2 has a winning strategy in the game
$$SG^{\beta}_{|\beta|}((P^{\mathcal{M}_{\beta,i}}, \tau^{\mathcal{M}_{\beta,i}}), (P^{\mathcal{M}_{\beta,i+1}}, \tau^{\mathcal{M}_{\beta,i+1}})).$$

Let λ be a cardinal large enough such that all the structures and winning strategies are in $H(\lambda)$. We now define a structure

$$\mathcal{A} = (H(\lambda), \in, \kappa, \dots)$$

inside which we code the structures $\mathcal{M}_{\beta,i}$ and the winning strategies, with the help of some large signature.

Let $\mathbb{M}$ be a ternary relation symbol, let P be a binary function symbol and let k be a unary function symbol. Define their interpretations in $\mathcal{A}$ as follows:

- $\mathbb{M}^{\mathcal{A}}_{\beta,i}(a) \quad :\Longleftrightarrow \quad a \in \mathcal{M}_{\beta,i}$.
- $P^{\mathcal{A}}(\beta, i) \ := \ \begin{cases} P^{\mathcal{M}_{\beta,i}} & \text{if } \beta \in \kappa \text{ and } i \leqslant k_\beta \\ \emptyset & \text{otherwise.}\end{cases}$

- $k_\beta^{\mathcal{A}} \;:=\; \begin{cases} k_\beta & \text{if } \beta \in \kappa. \\ \emptyset & \text{otherwise.} \end{cases}$

Let σ^+ be obtained from σ by adding two to the arity of each symbol. Define the interpretations for symbols $c_+, f_+, R_+ \in \sigma^+$ in such a way that they respect σ:

- $c_+^{\mathcal{A}}(\beta, i) \;:=\; \begin{cases} c^{\mathcal{M}_{\beta,i}} & \text{if } \beta \in \kappa \text{ and } i \leqslant k_\beta \\ \emptyset & \text{otherwise.} \end{cases}$

- $f_+^{\mathcal{A}}(\beta, i, \bar{a}) \;:=\; \begin{cases} f^{\mathcal{M}_{\beta,i}}(\bar{a}) & \text{if } \beta \in \kappa \text{ and } i \leqslant k_\beta \text{ and } \bar{a} \in \mathcal{M}_{\beta,i}^{|f|} \\ \emptyset & \text{otherwise.} \end{cases}$

- $R_+^{\mathcal{A}}(\beta, i, \bar{a}) \quad :\Longleftrightarrow \quad R^{\mathcal{M}_{\beta,i}}(\bar{a}).$

For each $\beta \in \kappa$ and $i < k_\beta$, let $\mathrm{st}_{\beta,i}$ be a set that codes a winning strategy for 2 in the game

$$SG^{\beta}_{|\beta|}((P^{\mathcal{M}_{\beta,i}}, \tau^{\mathcal{M}_{\beta,i}}), (P^{\mathcal{M}_{\beta,i+1}}, \tau^{\mathcal{M}_{\beta,i+1}})).$$

Let st be a ternary relation symbol and let

$$\mathrm{st}^{\mathcal{A}}_{\beta,i}(t) \quad :\Longleftrightarrow \quad t \in \mathrm{st}_{\beta,i}\,.$$

Then let φ be the conjunction of the following.

- A large enough finite fragment of ZFC together with the $\mathcal{L}_{\kappa\omega}$-sentence that expresses that there are no non-standard integers.
- Description of the models $\mathcal{M}_{\beta,i}$.
  - ”*For each β, k_β is a natural number, and if $\beta \notin \kappa$, then $k_\beta = \emptyset$.*”
  - ”$\forall x \quad x \in P_{\beta,i} \leftrightarrow P_+(\beta, i, x)$”
  - ”*Every $\mathbb{M}_{\beta,i}$ is $\sigma_{\beta,i}$-closed for $\beta \in \kappa$ and $i \leqslant k_\beta$*”.

    Being $\sigma_{\beta,i}$-closed is expressed by a conjunction over function and constant symbols in σ, of formulas of the form

    $$\begin{aligned} &\mathbb{M}(\beta, i, c_+(\beta, i)) \quad \text{and} \\ &\forall x_0, \ldots x_k (\bigwedge_{j \leqslant k} \mathbb{M}(\beta, i, x_j) \to \mathbb{M}(\beta, i, f_+(\beta, i, \bar{x})). \end{aligned}$$

  - ”*Every $(P_+)_{\beta,i}$ is $\tau_{\beta,i}$-closed, for $\beta \in \kappa$ and $i \leqslant k_\beta$*”

    This is similar.
- ”*For all $\beta \in \kappa$, $\mathbb{M}_{\beta,0} \models \psi$ and $\mathbb{M}_{\beta,k_\beta} \models \psi'$*”.

  This is possible since the logic $\mathcal{L}$ satisfies atomic substitution.

- *"For every $\beta \in \kappa$ and $i \in k_\beta$, the set* $\mathrm{st}_{\beta,i}$ *codes a winning strategy for* 2 *in the game*

$$SG^{\beta}_{|\beta|}((P^{\mathcal{M}_{\beta,i}}, \tau^{\mathcal{M}_{\beta,i}}), (P^{\mathcal{M}_{\beta,i+1}}, \tau^{\mathcal{M}_{\beta,i+1}}))\text{''}.$$

  This is expressed by writing down the following.

  1. *"Every $t \in \mathrm{st}_{\beta,i}$ is a triple*

     $$t = (\alpha_t, ht_t, g_t)$$

     such that $\alpha_t \in \beta$ is an ordinal, ht_t is a height function with domain included in $(P_+)_{\beta,i} \cup (P_+)_{\beta,i+1}$ and g_t is a partial isomorphism over τ from $(P_+)_{\beta,i}$ into $(P_+)_{\beta,i+1}$."

     This last condition on g_t is expressible by atomic substitution, by the formula

     $$\bigwedge_{\eta \text{ atomic } \tau\text{-formula}} \forall x_1, \dots x_k \in dom(g_t) \quad (P_+)_{\beta,i} \models \eta(\bar{x}) \leftrightarrow (P_+)_{\beta,i+1} \models \eta(\bar{x}).$$

  2. *"The starting state $(\beta, \emptyset, \emptyset)$ is in* $\mathrm{st}_{\beta,i}$*."*
  3. *"For every $t \in \mathrm{st}_{\beta,i}$, every $\alpha \in \alpha_t$ and every $A \in [P_{\beta,i}]^{\leqslant|\beta|} \cup [P_{\beta,i+1}]^{\leqslant|\beta|}$, there is $s \in \mathrm{st}_{\beta,i}$ that extends t, such that $\alpha_s = \alpha$ and $A \subseteq dom(ht_s)$."*

The sentence φ is an $\mathcal{L}$-sentence and it is satisfied in the structure $\mathcal{A}$. By the strong well-ordering number of the logic $\mathcal{L}$ being at most κ, there is a model $\mathcal{B}$ with some infinite $\beta \in \kappa^{\mathcal{B}}$ and some $\{B_i\}_{i\in\omega} \subseteq \mathcal{B}$ such that

1. $(\beta, \in^{\mathcal{B}})$ is ill-founded.
2. $\mathcal{B} \models \forall x \bigvee_{i\in\omega} x \in B_i$.
3. $\mathcal{B} \models \bigwedge_{i\in\omega} |B_i| \leqslant |\beta|$.

We derive a contradiction by showing that

$$((P_+)^{\mathcal{B}}_{\beta,0}, \tau^{\mathcal{B}}_{\beta,0}) \cong ((P_+)^{\mathcal{B}}_{\beta,k_\beta}, \tau^{\mathcal{B}}_{\beta,k_\beta}).$$

As the σ-structures $\mathbb{M}^{\mathcal{B}}_{\beta,0}$ and $\mathbb{M}^{\mathcal{B}}_{\beta,k_\beta}$ are models of ψ and ψ', respectively, we have that the restructions to τ of $(P_+)^{\mathcal{B}}_{\beta,0}$ and $(P_+)^{\mathcal{B}}_{\beta,k_\beta}$ are in $\mathcal{K}$ and $\mathcal{K}'$, respectively. The classes $\mathcal{K}$ and $\mathcal{K}'$ were assumed to be disjoint and closed under isomorphism, so this will be a contradiction.

Now, the model $\mathcal{B}$ thinks that there is a winning strategy for the player 2 in the game on $((P_+)^{\mathcal{B}}_{\beta,0}, \tau^{\mathcal{B}}_{\beta,0})$ and $((P_+)^{\mathcal{B}}_{\beta,k_\beta}, \tau^{\mathcal{B}}_{\beta,k_\beta})$, of height β and width $|\beta|^{\mathcal{B}}$. We describe a play for the player 1. The winning strategy for the player 2 gives the isomorphism.

The player 1 should play as follows:

- At odd steps $2i+1$: He picks an ill-founded ordinal and the set $B_i \cap P_{\beta,0}$.
- At even steps $2i+2$: He picks an ill-founded ordinal and the set $B_i \cap P_{\beta,k_\beta}$.

The player 2 does not lose because she has a winning strategy. However, the player 1 never falls to the ground, since he descends along an infinite descending sequence. As the model $\mathcal{B}$ has standard integers, every element the player 2 assigns a height to, will eventually reach the ground and she has to map it. Hence, she recursively builds a sequence

$$g_0 \subseteq g_1 \subseteq g_2 \subseteq \dots$$

of partial isomorphisms such that their union is an isomorphism

$$((P_+)^{\mathcal{B}}_{\beta,0}, \tau^{\mathcal{B}}_{\beta,0}) \cong ((P_+)^{\mathcal{B}}_{\beta,k_\beta}, \tau^{\mathcal{B}}_{\beta,k_\beta}).$$

This ends the proof.

□

We obtain as a corollary:

Theorem 4.7 (First Characterization Theorem, S. Shelah, [11]). *The logic $\mathcal{L}^1_\kappa$ is the maximal regular logic above $\mathcal{L}_{\kappa\omega}$ with occurrence number κ that has elementary submodel relations $\preccurlyeq_\psi$ with the Union Property and the Löwenheim-Skolem number below κ, for each sentence ψ.*

5 The logic $\mathcal{L}^1_\kappa$ from below

We prove that Shelah's logic $\mathcal{L}^1_\kappa$ is the minimal regular logic above $\mathcal{L}_{\kappa\omega}$ which is *strongly* Δ*-closed* and in which the so-called *countable* λ*-covering property* is definable for all $\lambda < \kappa$. We first discuss interpolation and define the strong Δ-closure, then we define the covering property, and finally prove this theorem.

This theorem together with the First characterization theorem 4.7 from the last section will give a full characterization of the logic $\mathcal{L}^1_\kappa$, as the unique regular logic above $\mathcal{L}_{\kappa\omega}$ with occurrence number κ which satisfies the following properties:

1. It has ψ-submodel relations $\preccurlyeq_\psi$ with the Union Property and Löwenheim-Skolem number strictly below κ, for each sentence ψ.
2. It is strongly Δ-closed, and the countable λ-covering property is definable, for every $\lambda < \kappa$.

Interpolation

The interpolation property has consequences on the definability of model classes. For example, thanks to interpolation, we will see that in the logic $\mathcal{L}^1_\kappa$, every class of sets of size λ, for any $\lambda < \kappa$, is definable in the empty signature. This shows that $\mathcal{L}^1_\kappa$ is a lot stronger than the logic $\mathcal{L}_{\kappa\omega}$, which cannot distinguish between any infinite cardinalities in the empty signature.

Let τ be a signature. Recall that a model class in the signature τ is projective if it is the projection of a definable class in a larger signature, i.e. it has the form

$$Mod_\tau(\exists\sigma\varphi) = \{\mathcal{M} \upharpoonright \tau : \mathcal{M} \models \varphi\}$$

for some signature σ and some $\tau \cup \sigma$-sentence φ.

Definition 5.1. A logic has *interpolation* if any two disjoint projective classes can be separated by a definable class.

We write $\varphi \models \psi$, where φ is a sentence in some signature τ and ψ is a sentence in some signature σ, if $Mod_{\tau\cup\sigma}(\varphi) \subseteq Mod_{\tau\cup\sigma}(\psi)$. The interpolation property has an equivalent formulation: *For any two sentences φ and φ' in some signatures τ_φ and $\tau_{\varphi'}$, such that $\varphi \models \varphi'$, there is a sentence ψ in the common signature $\tau_\varphi \cap \tau_{\varphi'}$ such that $\varphi \models \psi$ and $\psi \models \varphi'$.* Let us show that these two formulations are equivalent.

Lemma 5.2. *Let $\mathcal{L}$ be a logic. The following are equivalent:*

1. *Two disjoint projective classes can be separated by a definable class.*
2. *For any two sentences φ and φ' in some signatures τ_φ and $\tau_{\varphi'}$, such that $\varphi \models \varphi'$, there is a sentence ψ in the common signature $\tau_\varphi \cap \tau_{\varphi'}$ such that $\varphi \models \psi$ and $\psi \models \varphi'$.*

Proof. Let τ be a signature.

1. $\Rightarrow$ 2. Let σ and σ' be disjoint signatures. Assume that $\varphi \models \varphi'$, for some $\tau \cup \sigma$- and $\tau \cup \sigma'$-sentences φ and φ', respectively. Then $Mod(\exists\sigma\varphi)$ and $Mod(\exists\sigma'\neg\varphi')$ are disjoint projective τ-classes. By (1.) there is a τ-sentence ψ that separates them. This ψ is the interpolant.

2. $\Rightarrow$ 1. Conversely, suppose that $\exists\sigma\varphi$ and $\exists\sigma'\varphi'$ projectively define disjoint classes, for some $\tau\sigma$- and $\tau\sigma'$-sentences φ and φ', respectively. Then $\varphi \models \neg\varphi'$, and so by (2.) there is an interpolant ψ in the common signature. This ψ separates $\exists\sigma\varphi$ and $\exists\sigma'\varphi'$.

□

Theorem 5.3. *The logic $\mathcal{L}^1_\kappa$ has interpolation.*

Proof. Immediate corollary of proposition 4.6. □

Example: The logic $\mathcal{L}_{\kappa\omega}$ fails interpolation

The logic $\mathcal{L}_{\omega_1\omega}$ has interpolation (see for example [7]), but for $\kappa > \omega_1$, interpolation fails in the logic $\mathcal{L}_{\kappa\omega}$. Let us sketch an example of two $\mathcal{L}_{\kappa\omega}$-sentences, one implying the other, that do not have an interpolant in their common signature.

Let λ be a cardinal such that $\lambda^+ < \kappa$. Define two signatures

$$\sigma = \{c_i : i < \lambda\}$$
$$\sigma' = \{c_i : \lambda \leqslant i < \lambda^+\}.$$

The intersection of these signatures is empty. The σ-sentence, that expresses that every element is one of the constants,

$$\forall x \bigvee_{i<\lambda} x = c_i$$

projectively defines the class of sets of size at most λ. The σ'-sentence, that expresses that all constants are distinct,

$$\bigwedge_{\lambda\leqslant i<j<\lambda^+} c_i \neq c_j$$

projectively defines the class of sets of size at least λ^+. These classes are clearly disjoint. However, in the logic $\mathcal{L}_{\kappa\omega}$, any two infinite structures in the empty signature are elementarily equivalent, so there is no sentence in the signature $\sigma \cap \sigma'$ that would separate these classes, which shows that the interpolation fails for $\mathcal{L}_{\kappa\omega}$ whenever $\kappa > \omega_1$.

Notice that as the logic $\mathcal{L}^1_\kappa$ has interpolation and is above $\mathcal{L}_{\kappa\omega}$, it must contain a sentence separating these two sentences. Therefore, it must be powerful enough to distinguish between infinite cardinalities in the empty language.

In fact, full interpolation is not needed to separate the disjoint projective classes of sets of size at most λ and at least λ^+, from the above example. It would be enough to require complementary projective classes to be separable by a definable class.

Definition 5.4. A model class is a Δ-*class* if it is projective and co-projective. A logic is Δ-*closed* if every Δ-class is definable.

Of course, if a logic has interpolation, then it is Δ-closed. We also define the relativized version of the Δ-closure. Recall that a relativized projective class is the restriction of a projective class to some unary predicate, i.e. of the form

$$\{(P^{\mathcal{M}}, \tau^{\mathcal{M}}) : \mathcal{M} \models \varphi\},$$

for some signatures τ and σ, unary predicate $P \in \sigma \setminus \tau$ and some $\tau \cup \sigma$-sentence φ.

Definition 5.5. A class is a *relativized* Δ-*class* if it is relativized projective and its complement is relativized projective. A logic is *strongly* Δ-*closed* is every relativized Δ-class is definable.

Proposition 5.6. *The logic* $\mathcal{L}^1_\kappa$ *is strongly* Δ-*closed.*

Proof. Immediate corollary of the Proposition 4.6. $\square$

Covering

Example 5.7. Consider the real line $(\mathbb{R}, <)$ and let $\mathcal{I}$ be the set of open intervals. Consider the structure

$$(\mathbb{R} \cup \mathcal{I}, \mathbb{R}, \mathcal{I}, \in, <),$$

where $\mathbb{R}$ and $\mathcal{I}$ are understood as unary predicates, and $\in$ is the membership relation between the real numbers and the intervals. This structure has the so-called *countable covering property*, in the sense that every subset of $\mathbb{R}$ can be covered with countably many open intervals from $\mathcal{I}$.

Example 5.8. Let us work in the first-order logic for a moment, for the sake of this example. Assume that $\mathcal{M}$ is a structure in relational signature and λ is a cardinal. Let Σ be the set $\{N : \mathcal{N} \preccurlyeq \mathcal{M} \ \wedge \ |N| \leqslant \lambda\}$. Form the structure

$$\mathcal{S} := (\mathcal{M} \cup \Sigma, \mathcal{M}, \Sigma, E, \tau^{\mathcal{M}}),$$

where $\mathcal{M}$ and Σ are unary predicates with their natural interpretations, and

$$E(a, N) \iff a \in \mathcal{M} \ \wedge \ N \in \Sigma \ \wedge \ a \in N.$$

In particular, we have

$$\mathcal{N} = (\{a : E(a, N)\}, \tau^{\mathcal{M}}) \preccurlyeq \mathcal{M},$$

for each $N \in \Sigma$. This structure $\mathcal{S}$ has the *countable* λ-*covering property*, in the sense that every subset of $\mathcal{M}$ of size λ can be covered by countably many (in fact just one) elements from the set Σ.

The countable λ-covering property

Definition 5.9. Let E be a binary predicate symbol and let λ be a cardinal. An $\{E,\dots\}$-structure $\mathcal{N}$ has the *countable λ-covering property (with respect to the symbol E)* if every subset of the domain of the relation $E^{\mathcal{N}}$ of size at most λ can be covered by countably many elements of the form $P_E := \{a : E^{\mathcal{N}}(a, P)\}$, where $P \in \mathcal{N}$.

We say that the countable λ-covering property is definable in a logic if the class of $\{E\}$-structures having the countable λ-covering property is definable in that logic.

Observation 5.10. *In the logic $\mathcal{L}_{\kappa\omega}$, the countable ω_1-covering property is not definable.*

Proof. It suffices to exhibit two structures that are partially isomorphic, one of which has the countable ω_1-covering property and the other that does not have. Let $\mathcal{M}$ be the union of ω_1 together with its countable subsets. Let $\mathcal{N}$ be the union of ω_1 and its full powerset. Equip both sets with the relation $\in$, with its natural meaning. So we have

$$\mathcal{M} = (\omega_1 \cup [\omega_1]^{\leqslant\omega}, \in)$$
$$\mathcal{N} = (\omega_1 \cup \mathscr{P}(\omega_1), \in).$$

The resulting structures are partially isomorphic, thus $\mathcal{L}_{\kappa\omega}$-elementarily equivalent. The structure $\mathcal{N}$ of course has the countable ω_1-covering property. However, the structure $\mathcal{M}$ does not, as no uncountable set is included in a countable union of countable sets. □

Lemma 5.11. *In the logic $\mathcal{L}^1_\kappa$, the countable λ-covering property is definable, for every $\lambda < \kappa$.*

Proof. We show that the class of $\{E\}$-structures that have the countable λ-covering property is closed under the game of height $\omega + \omega + 1$ and width λ.

Let $\mathcal{M}$ and $\mathcal{N}$ be $\{E\}$-structures, such that $\mathcal{M}$ has the countable λ-covering property and suppose that the player 2 has a winning strategy in the game $SG^{\omega+\omega+1}_\lambda(\mathcal{M}, \mathcal{N})$. We show that the structure $\mathcal{N}$ has the countable λ-covering property as well.

Fix a winning strategy st for the player 2. Let $A \subseteq \mathcal{N}$ be a set of size $\leqslant \lambda$, such that $A \subseteq dom(E^{\mathcal{N}})$. We find a countable subset $D \subseteq \mathcal{N}$ such that

$$A \subseteq \bigcup_{n \in D} n^*$$

We do this by describing plays of the player 1. The winning strategy st will give the set D. Let the player 1 start by playing the set A and the ordinal $\omega+\omega$.

Let the player 2 reply according to the strategy st. Let $(A_i)_{i<\omega}$ be a partition of A, given by the height function chosen by the player 2, i.e., for each $i \in \omega$, $A_i = ht^{-1}\{i\}$.

For every $k \in \omega$, consider the following continuation of the game. The player 2 plays the empty set $\emptyset$ and the ordinal $\omega + k + 1$. The player 2 plays according to the strategy st, which is winning, so she must find a partial isomorphism

$$g_k : \bigcup_{i<k} A_i \to \mathcal{M},$$

before the player 2 hits the ordinal ω. Now, the structure $\mathcal{M}$ has the countable λ-covering property, so we may fix a countable set $B_k \subseteq \mathcal{M}$ such that

$$ran(g_k) \subseteq \bigcup_{m \in B_k} m^*.$$

Suppose that the player 1 plays the set B_k and the ordinal ω. Let the player 2 reply according to the strategy st - she obtains some partition $(B_k^j)_{j\in\omega}$ of the set B_k, given by her height function.

For every $l \in \omega$, consider the following continuation of the game. The player 1 plays the empty set $\emptyset$ and the natural number $l+1$. The player 2 plays according to the strategy st, and before the player 1 touches the height 0, she will have found a partial isomorphism $g_{k,l} \supseteq g_k$,

$$\bigcup_{i<l} B_k^i \subseteq ran(g_{k,l}).$$

Now, for each k and each l, let $D_{k,l}$ be the preimage of the set $\bigcup_{i<l} B_k^i$ under the partial isomorphism $g_{k,l}$. Then let $D := \bigcup_{i,k<\omega} D_i^k$. Clearly this set is countable. Furthermore, we have

$$A \subseteq \bigcup_{n \in D} n^*.$$

Hence the structure $\mathcal{N}$ also has the countable λ-covering property. $\square$

The Second Characterization Theorem for $\mathcal{L}^1_\kappa$

This last section is devoted to prove that the logic $\mathcal{L}^1_\kappa$ is the minimal regular logic above $\mathcal{L}_{\kappa\omega}$ with respect to the following properties:

1. Being strongly Δ-closed.

2. Being able to define the countable λ-covering property for every $\lambda < \kappa$.

Example: The ψ-models

Let ψ be some $\mathcal{L}^1_\kappa$-sentence and let γ be its Löwenheim-Skolem number. Recall that this means that for every structure $\mathcal{M}$, there is $\mathcal{N} \preccurlyeq_\psi \mathcal{M}$ of size γ. Let $\mathcal{M}$ be some model of ψ. Define a larger model, called a *ψ-model of $\mathcal{M}$*, in a signature augmented by unary predicate symbols $\mathcal{M}$ and Σ_ψ and a binary E, as follows:

- Let $\Sigma_\psi = \{N : \mathcal{N} \preccurlyeq_\psi \mathcal{M} \ \wedge\ |\mathcal{N}| \leqslant \gamma\}$. The domain of a ψ-model of $\mathcal{M}$ is the set $\mathcal{M} \cup \Sigma_\psi$.
- For two elements $a, N \in \mathcal{M}$, let
$$aEN \quad :\Longleftrightarrow \quad a \in \mathcal{M} \quad \wedge \quad N \in \Sigma_\psi \quad \wedge \quad a \in N.$$
- The interpretations of symbols of τ are equal to their interpretations in $\mathcal{M}$ on $\mathcal{M}$, and on the set Σ_ψ, they behave in whatever way.

A ψ-model of $\mathcal{M}$ has thus form
$$(\mathcal{M} \cup \Sigma_\psi, \mathcal{M}, \Sigma_\psi, E, \tau).$$
In particular, we have that for each $N \in \Sigma_\psi$,
$$N = \{a \in \mathcal{M} : aEN\}.$$

Also for the future: Notice that as every $m \in \Sigma_\psi$ has size at most γ, we could add a fourth symbol $<$ that codes a well-ordering of each $N \in \Sigma_\psi$, of order type at most γ.

Outline of the proof of the 2^{nd} Characterization Theorem

The goal is show that the logic $\mathcal{L}^1_\kappa$ is the minimal strongly Δ-closed regular logic above $\mathcal{L}_{\kappa\omega}$, in which the countable γ-covering is definable, for every $\gamma < \kappa$.

The proof runs as follows. We assume that $\mathcal{L}$ is a strongly Δ-closed regular logic above $\mathcal{L}_{\kappa\omega}$ in which the countable γ-covering is definable for every $\gamma < \kappa$. We need to show that $\mathcal{L}^1_\kappa \leqslant \mathcal{L}$, i.e., that every $\mathcal{L}^1_\kappa$-sentence is equivalent to an $\mathcal{L}$-sentence. As the logic $\mathcal{L}$ is strongly Δ-closed by assumption, it suffices to see that for every $\mathcal{L}^1_\kappa$-sentence ψ, the class $Mod_{\tau_\psi}(\psi)$, which is by definition equal to ψ itself, is a relativized projective class in $\mathcal{L}$. This suffices since it implies that both ψ and $\neg\psi$ are then relativized projective, for every $\mathcal{L}^1_\kappa$-sentence ψ, and hence definable, by the strong Δ-closure. We will thus find for every $\mathcal{L}^1_\kappa$-sentence ψ in some signature τ, an $\mathcal{L}$-sentence Φ_ψ, in an augmented signature, such that
$$\mathcal{M} \models \psi \iff \mathcal{M} \in \{(M^{\mathcal{N}}, \tau^{\mathcal{N}}) : \mathcal{N} \models \Phi_\psi\}.$$

For a fixed $\mathcal{L}^1_\kappa$-sentence ψ, the first task is to define the sentence Φ_ψ and the second task is to show that it indeed relatively projectively defines ψ.

The sentence Φ_ψ should have the property that in every model of it, the predicate M with the inherited τ-structure is a model of ψ. We will write down the sentence Φ_ψ so that in each model of it, the predicate Σ will consist of substructures of M that are models of ψ. Furthermore, the predicate Σ will be $\preccurlyeq_\psi$-directed. Finally, we will require that every model of Φ_ψ has a suitable covering property. This will be enough to imply that the predicate M is a model of ψ.

Defining the sentence Φ_ψ

Assume that $\mathcal{L}$ is a regular logic above the logic $\mathcal{L}_{\kappa\omega}$ in which the γ-covering property is definable for every cardinal $\gamma < \kappa$. Fix some $\mathcal{L}^1_\kappa$-sentence ψ. Let $\tau := \tau_\varphi$, and let γ be the Löwenheim-Skolem number of ψ. We will define an $\mathcal{L}$-sentence Φ_ψ in the signature $\tau \cup \{M, \Sigma, E, <\}$, where M and Σ are unary predicate symbols, E is a binary relation symbol and $<$ is a ternary relation symbol.

In an $\{E, \dots\}$-structure $\mathcal{N}$, for an element $P \in \mathcal{N}$, write

$$P_E := \{a : E^{\mathcal{N}}(a, P)\}$$
$$\Sigma^{\mathcal{N}}_E := \{(P_E, \tau^{\mathcal{N}}) : P \in \Sigma\}.$$

And in a $\{<, \dots\}$-structure $\mathcal{N}$, for elements $a, b, P \in \mathcal{M}$, write

$$a <_P b \quad :\Longleftrightarrow \quad <^{\mathcal{N}} (P, a, b).$$

The models of Φ_ψ will be the $\tau \cup \{M, \Sigma, E, <\}$-structures that satisfy the following.

1. *The predicate M is τ-closed.*
2. *For every $P \in \Sigma$, $(P_E, <_P)$ is a well ordering of order type $\leqslant \gamma$.*
3. *The set Σ_E is an $\preccurlyeq_\psi$-directed set of substructures of (M, τ).*
4. *Every structure from Σ_E is a model of ψ.*
5. *Every subset of the predicate M of size at most γ is contained in the union of countably many sets from Σ_E.*

The first four conditions will be definable in the logic $\mathcal{L}_{\kappa\omega}$. The last condition is just the γ-covering property.

Let us see how to express the properties *1.-4.*

Observation 5.12. *The first condition stating that the predicate M is τ-closed is expressible in $\mathcal{L}_{\kappa\omega}(\tau \cup \{M\})$, by the sentence*

$$\bigwedge_{f \in \tau \text{ function}} \forall \bar{x} \quad M(x_1) \wedge \dots \wedge M(x_k) \;\rightarrow\; M(f(\bar{x})) \quad \wedge \quad \bigwedge_{c \in \tau \text{ constant}} M(c).$$

Similarly, each P_E being τ-closed, for $P \in \Sigma$, is expressible in the logic $\mathcal{L}_{\kappa\omega}$.

Observation 5.13. *The second condition stating that $(P_E, <_P)$ is a well ordering of order type $\leqslant \gamma$, for each $P \in \Sigma$, is also expressible in the logic $\mathcal{L}_{\kappa\omega}$. This follows from the atomic substitution applied to the sentences axiomatizing well-orderings of type $\leqslant \gamma$, which are expressible in $\mathcal{L}_{\kappa\omega}$, as was shown in the Remark 2.6.*

Observation 5.14. *The third condition stating that the set Σ_E is an $\preccurlyeq_\psi$-directed set of substructures of (M, τ), is expressible in the logic $\mathcal{L}_{\kappa\omega}$.*

Proof. We show that there is an $\mathcal{L}_{\kappa\omega}$-formula in the variables x, y that expresses

> *"if x_E and y_E are τ-closed, and $(x_E, <_x)$ and $(y_E, <_y)$ are well-orders of type at most γ, then $x_E \preccurlyeq_\psi y_E$".*

The antecedent of the implication is expressible by the observations above. We write down a sentence that expresses the consequent "$x_E \preccurlyeq_{\lambda^+\theta^+} y_E$". The relation $\preccurlyeq_\psi$ is the relation $\preccurlyeq_{\lambda^+\theta^+}$ for some cardinals $\theta < \lambda < \kappa$, such that $\lambda = \beth_{\theta^+}$.

First we define for every $\mathcal{L}_{\lambda^+\theta^+}$-formula $\varphi(\bar{z})$ and every tuple $\bar{\alpha} \in \gamma^{\leqslant\theta}$ an $\mathcal{L}_{\kappa\omega}$-formula "$y_E \models \varphi(\bar{\alpha}^x)$" in two variables x and y. The intended meaning of this formula is the following: if the elements $a_1, \dots, a_i, \dots \in x_E$ have ranks $\alpha_1, \dots, \alpha_i, \dots$, respectively, in the well-order $(x_E, <_x)$, then $y_E \models \varphi(\bar{a})$.

Notice first that the formula *"z has rank α in the well-order $(x_E, <_x)$"* is defined by recursion exactly as in the Observation 2.5, but with $<$ replaced by the relation $<_x$.

The formula "$y_E \models \varphi(\bar{\alpha}^x)$" is defined as follows.

- For atomic formulas $\varphi(\bar{z})$ in k free variables and ordinals $\bar{\alpha} \in \gamma^k$, let
$$y_E \models \varphi(\bar{\alpha}^x) \quad :\equiv \quad \forall \bar{z}\Big(\bigwedge_{i<k} \text{"}z_i \textit{ has rank } \alpha_i \textit{ in } (x_E, <_x)\text{"} \;\to\; \varphi(\bar{z})\Big).$$

- For negation and conjunction and $\bar{\alpha} \in \gamma^{\leqslant\theta}$, let
$$y_E \models \neg\varphi(\bar{\alpha}^x) \quad :\equiv \quad \neg(y_E \models \varphi(\bar{\alpha}^x))$$
$$y_E \models \bigwedge_i \varphi_i(\bar{\alpha}^x) \quad :\equiv \quad \bigwedge_i (y_E \models \varphi_i(\bar{\alpha}^x))$$

- For $\exists \bar{w}\varphi(\bar{w}, \bar{z})$, let
$$y_E \models \exists \bar{w}\varphi(\bar{w}, \bar{\alpha}^x) \quad :\equiv \quad \bigvee_{\bar{\beta} \in \gamma^{\leqslant\theta}} y_E \models \varphi(\bar{\beta}^y, \bar{\alpha}^x)$$

Now the sentence
$$x^* \subseteq y^* \quad \wedge \quad \bigwedge_{\bar{\alpha} \in \gamma^{\leqslant\theta}} \; \bigwedge_{\varphi \in \mathcal{L}_{\lambda^+\theta^+}(\tau)} x^* \models \varphi(\bar{\alpha}^x) \leftrightarrow y^* \models \varphi(\bar{\alpha}^x)$$
defines "$x_E \preccurlyeq_\psi y_E$". This ends the proof. $\square$

Observation 5.15. *The fourth condition stating that every structure from Σ_E is a model of ψ, is expressible in the logic $\mathcal{L}_{\kappa\omega}$.*

Proof. We show that there is an $\mathcal{L}_{\kappa\omega}$-formula that expresses

> *"if x_E is τ-closed, and $(x_E, <_x)$ is a well-order of order type at most γ, then $(x_E, \tau) \models \psi$".*

The antecedent of the implication is expressible by the observations above. We express the consequent "$(x_E, \tau) \models \psi$".

Let K_ψ be the set of models of ψ that have their domain subset of γ. In other words, let

$$K_\psi \ := \ \{(P, \in, \tau^P) \ : \ (P, \tau^P) \models \psi \ \wedge \ P \subseteq \gamma \ \}.$$

Now, if $(x_E, <_x)$ is a well-ordering of order type $\leqslant \gamma$, then $(x_E, \tau) \models \psi$ if and only if $(x_E, <_x, \tau)$ is isomorphic to some model in K_ψ. By the Remark 2.7 in the first section, for every $\mathcal{P} = (P, \in, \tau^P) \in K_\psi$, there is an $\mathcal{L}_{\kappa\omega}(\tau \cup \{<\})$-sentence $\varphi_{\mathcal{P}}$ that characterizes $\mathcal{P}$ up to isomorphism. Hence "$(x_E, \tau) \models \psi$" is expressed by

$$\bigvee_{\mathcal{P} \in K_\psi} (x_E, <_x, \tau) \models \varphi_{\mathcal{P}}.$$

This is an $\mathcal{L}_{\kappa\omega}$-sentence, by the atomic substitution property.

□

Recap. By the Observations above, for every $\mathcal{L}^1_\kappa$-sentence ψ in signature $\tau := \tau_\psi$ and with Löwenheim-Skolem number γ, there is an $\mathcal{L}_{\kappa\omega}$-sentence Φ_ψ in the signature $\tau \cup \{M, \Sigma, E, <\}$, that satisfies

1. *The predicate M is τ-closed.*
2. *For every $P \in \Sigma$, $(P_E, <_P)$ is a well ordering of order type $\leqslant \gamma$.*
3. *The set Σ_E is an $\preccurlyeq_\psi$-directed set of substructures of (M, τ).*
4. *Every structure from Σ_E is a model of ψ.*
5. *Every subset of M of size at most γ is contained in the union of countably many sets from Σ_E.*

Observation 5.16. *Let $\mathcal{M}$ be an arbitrary model of ψ. Let*

$$\mathcal{N} := (\mathcal{M} \cup \Sigma, \mathcal{M}, \Sigma, E, <, \tau),$$

where $\Sigma_E^{\mathcal{N}}$ is the set of $\preccurlyeq_\psi$-elementary submodels of $\mathcal{M}$ of size at most γ, $<$ well-orders every model in Σ and $E(a, P)$ denotes the real membership restricted to the case where $a \in \mathcal{M}$ and $P \in \Sigma$. This $\mathcal{N}$ clearly satisfies Φ_ψ.

We have proved the following.

Proposition 5.17. *Assume that $\mathcal{L}$ is a regular logic above $\mathcal{L}_{\kappa\omega}$, strong enough to express the countable γ-covering property for any $\gamma < \kappa$. For every $\mathcal{L}^1_\kappa$-sentence ψ with Löwenheim-Skolem number γ, there is an $\mathcal{L}$-sentence Φ_ψ in the signature $\tau_\psi \cup \{M, \Sigma, E, <\}$ such that*

1. *Every model of ψ has an extended expansion into a model of Φ_ψ.*
2. *Every model of Φ_ψ satisfies the following properties:*
   - (a) *The set Σ_E is an $\preccurlyeq_\psi$-directed set of substructures of (M, τ_ψ), each of which has size at most γ and is a model of ψ.*
   - (b) *Every subset of M of size at most γ is contained in the union of countably many sets from Σ_E.*

The sentence Φ_ψ relatively projectively defines ψ

In order to finish the proof that $\mathcal{L}^1_\kappa$ is minimal, we still have to see that the sentence Φ_ψ relatively projectively defines ψ.

Proposition 5.18. *Assume that $\mathcal{L}$ is a regular logic above $\mathcal{L}_{\kappa\omega}$ in which the countable γ-covering is definable for every $\gamma < \kappa$. Every $\mathcal{L}^1_\kappa$-sentence is a relativized projective class in $\mathcal{L}$.*

Proof. Let ψ be some $\mathcal{L}^1_\kappa$-sentence, let $\tau := \tau_\psi$ and let γ be the Löwenheim-Skolem number of ψ. Let Φ_ψ be the sentence from the Proposition 5.17. We show that Φ_ψ relatively projectively defines ψ, i.e. that

$$\mathcal{M} \models \psi \iff \mathcal{M} \in \{(M^{\mathcal{N}}, \tau^{\mathcal{N}}) : \mathcal{N} \models \Phi_\psi\}.$$

The direction "$\Rightarrow$" is true because every model of ψ has an expanded extension into a model of Φ_ψ. For the direction "$\Leftarrow$", assume that $\mathcal{N} \models \Phi_\psi$ for some structure $\mathcal{N}$, and let $\mathcal{M} = (M^{\mathcal{N}}, \tau^{\mathcal{N}})$. We show that $\mathcal{M}$ satisfies the sentence ψ. Suppose to the contrary that $\mathcal{M} \models \neg\psi$.

We will derive a contradiction by recursively defining two $\preccurlyeq_\psi$-chains $(A_i)_i$ and $(B_i)_i$ such that the A_i's are models of $\neg\psi$ and the B_i's are models of ψ, but

$$\bigcup_i A_i = \bigcup_i B_i.$$

This will be a contradiction because the relation $\preccurlyeq_\psi$ satisfies the Union Property.

Recall that the set $\Sigma_E^{\mathcal{N}}$ is an $\preccurlyeq_\psi$-directed set of substructures of $\mathcal{M}$, each of which has size at most γ and is a model of ψ and every subset of M of size at most γ is contained in the union of countably many sets of the form P_E, where $P \in \mathcal{N}$.

Step 0: Choose any $A_0 \preccurlyeq_\psi \mathcal{M}$ of size at most γ. Now $A_0 \models \neg\psi$, because we assumed that $\mathcal{M} \models \neg\psi$. As the model $\mathcal{N}$ has the countable γ-covering property, there are $B^0_j \in \mathcal{N}$, $j \in \omega$, such that

$$A_0 \subseteq \bigcup_{j\in\omega} (B^0_j)_E$$

Every $((B^0_j)_E, \tau^{\mathcal{N}})$ is a model of ψ, of size at most γ. Let B_0 be $((B^0_0)_E, \tau^{\mathcal{N}})$.

Step n: Assume that the models A_k, B_k and the sets $(B^k_j)_{j\in\omega}$ were built, for each $k < n$. They all have size at most γ.

- Let $A_n \preccurlyeq_\psi \mathcal{M}$ be of size at most γ, such that for each $k < n$, $A_k, B_k \subseteq A_n$.
- By the covering property, there are $B^n_j \in \Sigma$, $j \in \omega$ such that

$$A_n \subseteq \bigcup_{j\in\omega} (B^n_j)_E.$$

As the set $\Sigma^{\mathcal{N}}_E$ is $\preccurlyeq_\psi$-directed, we may choose $B_n \in \Sigma^{\mathcal{N}}_E$ such that for each $k < n$, it contains B_k and for all $j, k \leqslant n$, it contains each B^k_j, in other words

$$\bigcup_{k<n} B_k \cup \bigcup_{j,k\leqslant n} B^k_j \subseteq B_n.$$

Now $(A_i)_i$ is an $\preccurlyeq_\psi$-chain by the Tarski-Vaught property of the relation $\preccurlyeq_\psi$, and $(B_i)_i$ is an $\preccurlyeq_\psi$-chain by the fact that each B_i was chosen from $\Sigma^{\mathcal{N}}_E$. Let

$$\mathcal{A} := \bigcup_i A_i \quad \text{and} \quad \mathcal{B} := \bigcup_i B_i.$$

By the Union Lemma 3.19, $\mathcal{A} \models \neg\psi$ and $\mathcal{B} \models \psi$.

We show that $\mathcal{A} = \mathcal{B}$. To see $\mathcal{A} \subseteq \mathcal{B}$: if $a \in \mathcal{A}$, then there is n such that $a \in A_n$, and thus $a \in B^n_j$ for some $j \in \omega$. Then $a \in B_m$ for any $m \geqslant max\{n, j\}$. Thus $\mathcal{A} \subseteq \mathcal{B}$. The direction $\mathcal{B} \subseteq \mathcal{A}$ is clear. This is a contradiction and ends the proof.

$\square$

We now have as a corollary:

Theorem 5.19 (The 2^{nd} Characterization Theorem)**.** *The logic $\mathcal{L}^1_\kappa$ is the minimal logic among the regular logics above $\mathcal{L}_{\kappa\omega}$ that are strongly Δ-closed and in which the countable γ-covering is definable for all $\gamma < \kappa$.*

Proof. Let $\mathcal{L}$ be a logic above $\mathcal{L}_{\kappa\omega}$ satisfying the required properties. For every $\mathcal{L}^1_\kappa$-sentence ψ, there is an $\mathcal{L}$-sentence Φ_ψ, that witnesses that ψ is relativized projective in $\mathcal{L}$. As $\mathcal{L}$ is strongly Δ-closed and both ψ and $\neg\psi$ are relativized projective, it follows that both ψ and $\neg\psi$ are definable in $\mathcal{L}$. $\square$

By putting the First and the Second Characterization Theorems together, we get the following.

Theorem 5.20 (Characterization of $\mathcal{L}^1_\kappa$)**.** *The logic $\mathcal{L}^1_\kappa$ is the unique regular logic above $\mathcal{L}_{\kappa\omega}$ with occurrence number κ that satisfies the following properties.*

1. *It has ψ-submodel relations $\preccurlyeq_\psi$ with the Union Property and Löwenheim-Skolem number strictly below κ, for each sentence ψ.*
2. *The countable λ-covering property is definable, for every $\lambda < \kappa$.*
3. *It is strongly Δ-closed.*

6 Conclusion: Is $\mathcal{L}^1_\kappa$ a logic?

In the introduction, the question was raised, whether $\mathcal{L}^1_\kappa$, despite having no known recursive syntax, can count as a logic. The trivial answer is yes, $\mathcal{L}^1_\kappa$ is a logic, because it satisfies the definition of a logic. Therefore, we are led to ask whether Lindström's definition of an abstract logic is a good definition, i.e., if it manages to capture sufficient conditions for being a logic in the non-technical sense.

Without known syntax, the logic $\mathcal{L}^1_\kappa$ indeed is quite pathological. However, as mentioned, there are indications that it has a well-behaved model theory. The first indication is the Lindström characterization and the interpolation. Another one is the weak compactness property, *the strong well-ordering number* κ, defined in the third chapter. In addition to these, Shelah proved in [12] an algebraic characterization for $\mathcal{L}^1_\kappa$. Analogously to the Keisler-Shelah characterization of first-order elementary equivalence according to which two structures are (first order) elementarily equivalent if and only if they have isomorphic ultrapowers, in the case of the logic $\mathcal{L}^1_\kappa$, two structures are $\mathcal{L}^1_\kappa$-equivalent if and only if they have isomorphic iterated ultrapowers. This has a corollary, also found in [12], that every complete $\mathcal{L}^1_\kappa$-theory has a so-called *special model*. A special model is a weak version of a saturated model.

So far we argued that $\mathcal{L}^1_\kappa$ should count as a logic, because it seems to have a well behaved model theory. This is a very utilitarian justification. A logician wants a logic with good model theory, because then she has a well-equipped toolpack to prove beautiful theorems. One may also try to find another kind of justifications.

An abstract model theorist may ask the following question: Given a collection of model classes, what is a *natural* logic for it? A natural logic should be powerful enough to distinguish between two models from two disjoint model classes but still "tame enough" to have useful model-theoretic properties. One could try to argue that $\mathcal{L}^1_\kappa$ is a natural logic of some mathematical phenomenon, and therefore justifiably a logic, even without recursive syntax. Each $\mathcal{L}^1_\kappa$-sentence ψ together with its relation $\preccurlyeq_\psi$ is an abstract elementary class, and A. Villaveces has asked if the logic $\mathcal{L}^1_\kappa$ could be a natural logic for abstract elementary classes. Some discussion is in [13]. A positive answer to this question would indeed justify that the logic $\mathcal{L}^1_\kappa$ is a logic.

A third kind of justification would be, of course, in the form of critizism of critizism. The only objection why the logic $\mathcal{L}^1_\kappa$ would not be taken seriously as a logic, by someone who believes that the standard infinitary logics $\mathcal{L}_{\kappa\lambda}$ are justifiably logics, is the lack of recursive syntax. Therefore the question is: to what extent is a recursive syntax essential to a logic? In Shelah's logic, the sentences are given in terms of a game, rather than recursive syntax. The situation is very analogous to natural language, in which the meanings of words and expressions can be thought to be determined by a language game, in the wittgensteinian sense. The meaning of a word is determined in its use: a speaker "wins a round" if she manages to get understood by the hearer. So why would a formal language be less natural than the natural language, if it were defined in terms of a game? But now there is a critizism of critizism of critizism: Perhaps it would not be less natural, but it might be less useful. Namely, the lack of recursive syntax has the practical annoying consequence that there are no more proofs by induction on the complexity of a formula.

Whatever one decides on the nature of $\mathcal{L}^1_\kappa$, it cannot be denied that as the only known infinitary logic having both the interpolation property and a Lindström theorem, it provides important insight on the structure of the hierarchy of abstract logics.

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