Siiri Kivimäki

On universality problems in non-elementary classes

Doctoral dissertation

To be presented for public discussion with the permissions of the Faculties of Science of the University of Helsinki and Université Paris Cité, in Porthania, Suomen laki -sali, Yliopistonkatu 3, Helsinki, on the 16th of May 2026 at 12 o'clock.

This dissertation was prepared in cotutelle between Department of Mathematics and Statistics, Faculty of Science, University of Helsinki and Institut de Mathématiques de Jussieu-Paris Rive Gauche, Université Paris Cité.

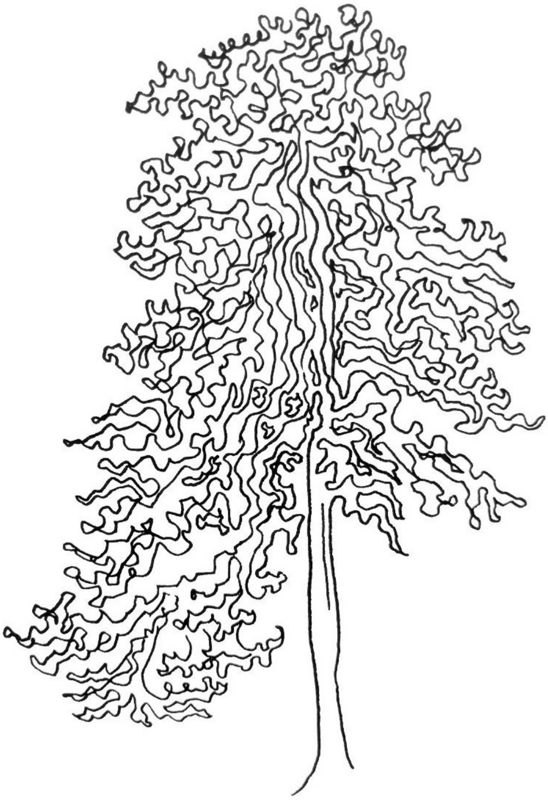
Supervisors (Directeurs de thèse)	Prof. Juliette Kennedy, University of Helsinki Prof. Jouko Väänänen, University of Helsinki Prof. Boban Veličković, Université Paris Cité
Coordinating academic (Président)	Prof. Juha Kontinen, University of Helsinki
Custos	Prof. Jouko Väänänen, University of Helsinki
Opponent	Prof. Sandra Müller, Technische Universität Wien
Pre-examiners (Rapporteurs)	Prof. Sandra Müller, Technische Universität Wien Prof. Chris Lambie-Hanson, Czech Academy of Sciences
Academic jury (Examinatrices)	Prof. Natasha Dobrinen, University of Notre Dame Prof. Vera Fischer, University of Vienna

Publisher: Helsingin yliopisto
Series: Dissertationes Universitatis Helsingiensis 243/2026

ISBN 978-952-84-2360-7 (pehmeäkantinen/paperback)
ISBN 978-952-84-2359-1 (PDF)
ISSN 2954-2898 (painettu/print)
ISSN 2954-2952 (verkkojulkaisu/online)
PunaMusta, Joensuu 2026

Meillä kasvo suuri tammi, yleni emopetäjä;
Ei uo tammen tagrojaa, puun pitkän lyhentäijää.
Ent voint risulta kävvä, tammen oksilta tallaella.

Suomen kansan vanhat runot III1 615



Tiivistelmä

Olkoon $\mathcal{K}$ kokoelma alkioita, joiden välille on määritelty upotus. Sellaista $\mathcal{K}$:n alkiota, johon $\mathcal{K}$:n jokainen alkio uppoaa, kutsutaan $\mathcal{K}$:n *universaaliksi alkioksi.* Universaalin alkion olemassaolo on usein riippumatonta joukko-opin aksioomista – erityisesti silloin, kun luokan alkiot ovat ylinumeroituvia.

Tässä työssä esitetään ensin metodi universaalien alkioiden olemassaolon ristiriidattomuuden todistamiseksi ja sitten tätä metodia sovelletaan kolmeen luokkaan: numeroituvasti kromaattisiin verkkoihin, eräisiin puihin ja tietynlaisiin lineaarijärjestyksiin. Jokainen näistä luokista on *epäelementaarinen*, siis ei määriteltävissä ensimmäisen kertaluvun kaavalla.

Ensimmäisen luokan alkiot ovat pareja (G, E, c), missä (G, E) on $\aleph_1$:n kokoinen verkko ja $c : G \to \mathbb{N}$ on väritys. Upotukset tällaisten parien välillä ovat värin säilyttäviä verkko-homomorfismeja. Toinen luokka koostuu nk. *leveistä Aronszajnin κ-puista.* Nämä ovat κ:n kokoisia puita, joista ei löydy κ:n pituista oksaa. Kardinaali κ on joko $\aleph_1$ tai ylinumeroituva kaksoisseuraajakardinaali. Upotukset ovat injektiivisiä korkeuden säilyttäviä puu-upotuksia. Kolmas luokka koostuu sellaisista κ:n kokoisista lineaarijärjestyksistä, jotka eivät sisällä järjestysisomorfista kopiota κ:sta tai sen käänteisestä järjestyksestä κ^{-1}:stä. Upotukset tässä luokassa ovat järjestyksen säilyttäviä funktioita. Tässäkin tapauksessa κ on joko $\aleph_1$ tai kaksoisseuraajakardinaali.

Itse metodi on pakotusteoreettinen, ja pakotuskonstruktioissa käytetään pieniä ZFC^-:n malleja sivuehtoina. Metodilla saadaan kullekin yllämainitulle luokalle pakotuslaajennos, jossa luokka sisältää universaalin alkion, ja kombinatorinen väite $2^{<\kappa} > \kappa^+$ (ja verkkojen tapauksessa $2^\omega > \aleph_1$) pätee. Kahden jälkimmäisen luokan – puiden ja lineaarijärjestysten – universaalin alkion olemassaolon ristiriidattomuustodistuksissa käytetään heikosti kompaktia kardinaalia.

Résumé

Soit $\mathcal{K}$ une classe munie d'une notion de plongement. Un objet $a \in \mathcal{K}$ est dit *universel* si tout objet de $\mathcal{K}$ se plonge dans a. L'existence d'un tel objet est souvent indépendante des axiomes de la théorie des ensembles, surtout si la classe $\mathcal{K}$ contient des objets non dénombrables.

Dans cette thèse, une méthode permettant de prouver la cohérence de l'existence d'un objet universel est présentée, puis appliquée à trois cas : certains graphes, certains arbres et certains ordres linéaires. Chacune de ces classes est *non-élémentaire*, ce qui veut dire qu'elle n'est pas définissable par une formule du premier ordre.

Plus précisément, les objets de la première classe sont des graphes de taille $\aleph_1$ munis d'un colorage dans un nombre dénombrable de couleurs. Les plongements sont des homomorphismes de graphes préservant la couleur. La deuxième classe consiste en *κ-arbres d'Aronszajn larges*, c'est-à-dire des arbres de taille κ sans suite de taille κ ordonnée linéairement par l'ordre de l'arbre. Chez ces arbres, les plongements sont des plongements d'arbre injectifs qui préserve la hauteur des nœuds. Le cardinal κ est soit $\aleph_1$, soit un cardinal non-dénombrable de la forme μ^{++}. Les objets de la troisième classe sont des ordres linéaires de taille κ qui ne contiennent pas de partie isomorphe à κ ou à son inverse, κ^{-1}. Le cardinal κ est également soit $\aleph_1$, soit un successeur double.

La méthode en question est basée sur la théorie du forçage itéré. Elle utilise des petits modèles de ZFC^{-1} comme conditions latérales. L'extension générique obtenu satisfera $2^{<\kappa} > \kappa^+$ (ou $2^\omega > \aleph_1$ au cas des graphes). Les preuves, à l'exception de la cohérence de l'existence d'un graphe universel dénombrablement chromatique de taille $\aleph_1$, supposent l'existence d'un cardinal faiblement compact.

Abstract

A *universality question* asks: given a class $\mathcal{K}$ equipped with a notion of embedding, is there an object $a \in \mathcal{K}$ such that every object in $\mathcal{K}$ embeds into a. Such an object is called *universal* in $\mathcal{K}$. The existence of a universal object is very often independent of the axioms of set theory among classes that consist of uncountable objects.

In this thesis, we present a method for proving the consistency of the existence of a universal object. We apply the method in various cases, including a certain class of graphs of bounded chromatic number, a certain class of trees, and a certain class of linear orders. Each of these classes is *non-elementary* in the sense that it is not definable by a first-order formula.

Specifically, the first class consists of graphs of size $\aleph_1$ with a coloring map into countably many colors, equipped with color-preserving graph-homomorphisms. The second class consists of *wide κ-Aronszajn trees* equipped with injective and level-preserving tree-embeddings. These are trees of size κ that do not contain a linearly ordered subset of size κ. The cardinal κ can be either $\aleph_1$ or any double successor cardinal. The third class is the class of linear orders of size κ that do not contain an order-isomorphic copy of κ nor its inverse κ^{-1}, equipped with order-preserving maps. Again κ can be either $\aleph_1$ or any double successor cardinal.

The method in question is based on the theory of iterated forcing, and relies on the use of small models of ZFC^- as side-conditions. The obtained generic extension will satisfy $2^{<\kappa} > \kappa^+$ (or $2^\omega > \aleph_1$ in the case of graphs). The proofs, except for the consistency of the existence of a universal countably chromatic graph of size $\aleph_1$, involve the existence of a weakly compact cardinal.

Acknowledgements

I am indebted to all my teachers, colleagues, thesis siblings, friends and family. It is impossible to thank enough my advisors Professors Juliette Kennedy, Jouko Väänänen and Boban Veličković, and equally importantly Professors Omer Ben-Neria and Menachem Magidor, with whom I had the honor to work during my doctoral studies. I consider myself extremely lucky to have been able to learn from these five amazing mathematicians. They taught me how to think, read and write.

Results in Chapter 6 were proved in collaboration with Professors Ben-Neria, Magidor and Väänänen. I owe a lot to them a lot in other chapters too. Results in Chapter 7 were proved in collaboration with Haytham Hammud, under the supervision of Professor Ben-Neria. Most of the results of the thesis are available online in three arXiv -preprints [12], [13], [1].

Contents

Chapter 1

Introduction

According to Finnish mythology, a thing is mastered with its “words of creation” (*syntysanat*). By knowing the words of creation of skies, wind can be sang into sails. In old tales, by singing, ships are built, swamplands are created, and trees are grown — even a spruce so tall that the moon and Ursa Major rest on its branches. In this thesis, we use the words of creation to grow a universal tree: an enormous tree that contains all the other trees.

1.1 Universality problems

A *universality problem* is a problem of the form

Given a class $(\mathcal{K}, \hookrightarrow)$*, does it contain a universal object?*

The class $\mathcal{K}$ can be any class of structures, for instance the class of countable groups, databases, or metric spaces, and $\hookrightarrow$ is any notion of embedding between them, for example homomorphism. A *universal object* in $\mathcal{K}$ is an object $a \in \mathcal{K}$ such that every $b \in \mathcal{K}$ embeds into a. Universality problems arise everywhere in mathematics and there exist various known methods for proving the existence of a universal object, whenever one exists. Two of the classical examples of universal objects are the rational numbers in the class of countable linear orders, equipped with order preserving maps, and the random graph in the class of countable graphs, equipped with graph homomorphism.

The picture changes radically when the objects are allowed to be uncountable. In this case, the existence of a universal object is often independent of the axioms of ZFC.

For instance, both the existence of a universal linear order of size $\aleph_1$ and the existence of a universal graph of size $\aleph_1$ are independent of set theory. Under the continuum hypothesis the theories of dense linear orders and random graph both have a saturated model, which are universal. This is a classical theorem in model theory (see [25] for details). The consistency of non-existence of a universal linear order or graph follow by adding $\aleph_2$ many Cohen reals. In fact, after adding $\aleph_2$ many Cohen reals, no unstable theory has a universal model in cardinality $\aleph_1$. In fact this is true for higher cardinals too, and for any *elementary class*, a class of the form $\mathsf{Mod}_\kappa(T) = \{M : M \models T \text{ and } |M| \leqslant \kappa\}$ for a complete first-order theory T equipped with elementary embeddings. See the appendix of [14] for a proof.

In case of elementary classes, the "canonical candidate" for a universal object is a *saturated model*, which is always universal. To prove the existence of a saturated model is easier than to prove the existence of a universal model, since being saturated is a property of one model, whereas being universal is a property of the whole class. Indeed, there exist several methods for producing a saturated model that work whenever one exists (realizing types one by one, taking an ultrapower,...). However, the existence of a saturated model of a theory T is equivalent to the statement ``$\kappa^{<\kappa} = \kappa$", whenever $T$ is unstable in $\kappa$ (see Theorem 4.7 on page 476 in [22]). And this combinatorial statement ``$\kappa^{<\kappa} = \kappa$" is independent from ZFC, for any uncountable cardinal κ. No such combinatorial characterization is known for the existence of universal models, which is often strictly weaker than the existence of saturated models. Shelah showed in [24] that there can exist a universal graph of size $\aleph_1$, even without a saturated one (i.e. without the continuum hypothesis). This is an example of a scenario where a universal model of size $\aleph_1$ does exist, despite the fact that a saturated model of size $\aleph_1$ cannot not exist by the failure of CH (in the case of graphs the relevant complete first-order theory is the theory of random graph). Not many other examples are known.

In [3], Shelah and Džamonja study properties of theories that would preclude the existence of a universal model whenever a saturated model does not exist. They conjecture that every NSOP_4-theory (see [25] for definition) has a universal model of size $\aleph_1$ in a ccc forcing extension that satisfies $\neg$CH. This conjecture is wide open. It even remains open whether there can exist a universal linear order of size $\aleph_1$ with $\neg$CH, or a universal triangle-free graph of size $\aleph_1$, also with $\neg$CH. The theory of triangle-free random graphs is NSOP_4, so solving their conjecture would require solving the case of triangle-free graphs.

In [17], Mekler provides a general method for producing a model of set theory with a universal object of size $\aleph_1$ in a certain type of an amalgamation class, together with $\neg$CH. His approach suggests that a shift of attention away from elementary classes to classes with simpler morphisms could provide useful in solving universality problems at uncountable cardinalities. This thesis considers universality problems in some non-elementary classes consisting of uncountable objects. Namely, in a certain class of trees, in a certain class of linear orders and in a certain class of graphs.

A *tree* is a partial order such that the set of predecessors of each element is well-ordered. A *branch* in a tree is a linearly ordered subset. Denote

$$\mathcal{T}_\kappa := \text{ trees of size } \kappa \text{ with no branch of length } \kappa.$$

An embedding between two trees in $\mathcal{T}_\kappa$ is a map that preserves levels and the tree order. Chapter 6 presents a proof of how to force a universal object into $\mathcal{T}_{\aleph_2}$, or more generally into $\mathcal{T}_{\mu^+}$ where μ is an uncountable successor cardinal.

Theorem 6.3.11. *Let μ be an uncountable successor cardinal. Assuming the consistency of a weakly compact cardinal*[1]*, consistently, there is a universal object in the class $\mathcal{T}_{\mu^+}$.*

The proof is a result of a collaboration with Professors Omer Ben-Neria, Menachem Magidor and Jouko Väänänen.

Contrary to the case of elementary classes where the hypothesis $\kappa^{<\kappa} = \kappa$ implies the existence of a saturated, hence universal, object, in the case of trees, the hypothesis $\kappa^{<\kappa} = \kappa$ implies that there is *no* universal object in $\mathcal{T}_\kappa$. This is due to Kurepa [15]. Indeed, in our model we have $2^\mu = \mu^{++}$. In Chapter 5, this result is adapted to the case of $\aleph_1$:

Theorem 5.3.12. *Assuming the consistency of a weakly compact cardinal, consistently, there is a universal object in the class $\mathcal{T}_{\aleph_1}$.*

The proof of Theorem 5.3.12 differs from the proof of Theorem 6.3.11 sin that the forcing poset in question consists of finite conditions, and therefore has no closure properties. This results in an extra complication in the proof of preservation of $\aleph_1$. On the other hand, in other places the proof becomes simpler, as the support of conditions is also finite. By Kurepa's result, the existence of a universal object in $\mathcal{T}_{\aleph_1}$ must violate the continuum hypothesis, and indeed, $2^{\omega_1} = \aleph_3$ holds in the generic extension of Theorem 5.3.12. In addition to Kurepa's result, the case of $\mathcal{T}_{\aleph_1}$ contrasts with a result by Džamonja and Shelah [4] by which $\mathsf{MA}_{\aleph_1}$ implies that $\mathcal{T}_{\aleph_1}$ does not contain a universal object.

Theorems 6.3.11 and 5.3.12 leave open whether $\mathcal{T}_\kappa$ can have a universal object when κ is a successor of a singular.

Question 1.1.1. *Can $\mathcal{T}_\kappa$ have a universal object when κ is a successor of a singular cardinal?*

---

[1] The existence of a weakly compact cardinal is a mild strengthening of the axioms of set theory

Question 1.1.1 is potentially hard and would require combining different techniques.

Next we consider universality question among linear orders. An embedding between two linear orders is an order-preserving map. If $2^\kappa = \kappa^+$, there is a saturated dense linear order of size κ, which is universal for linear orders of the same size, as the class of linear orders is an elementary class. However, it is a longstanding open question whether there can be one without the assumption about cardinal arithmetic.

Question 1.1.2. *Let $\kappa \geqslant \aleph_0$. Is is consistent that there is a universal linear order of size κ^+ and $2^\kappa > \kappa^+$?*

For $\aleph_1$, the question asks whether the existence of a universal linear order of size $\aleph_1$ together with the failure of continuum hypothesis is consistent. Consider the following subclass of linear orders:

$$\mathcal{L}_\kappa := \text{the class of linear orders of size } \kappa \text{ that do not order-embed } \kappa \text{ nor } \kappa^{-1}.$$

For instance, every subset of the real line $\mathbb{R}$ of size $\aleph_1$ belongs to $\mathcal{L}_{\aleph_1}$. In Chapter 7, we solve the universality problem in $\mathcal{L}_\kappa$:

Theorem 7.3.20. *Let $\mu \in \{\aleph_0\} \cup \{\lambda^+ : \lambda \in \mathsf{Card}\}$. Assuming the existence of a weakly compact cardinal, it is consistent that $\mathcal{L}_{\mu^+}$ has a universal object.*

The proof builds on [1] and [13] and results from a collaboration with Haytham Hammud.

In [19], Moore proved that PFA implies the existence of a universal *Aronszajn line*, i.e. a linear order of size $\aleph_1$ that does not contain an order-isomorphic copy of ω_1 nor its inverse ω_1^{-1}, nor an uncountable subset of the real numbers. Aronszajn lines thus form a proper subclass of $\mathcal{L}_{\aleph_1}$. It should be noted that the consistency strength of PFA is much more than a weakly compact cardinal, and it is not known what is the consistency stregth of the existence of a universal Aronszajn line. Our proof of Theorem 7.3.20 as such, however, does not generalize to produce a universal Aronszajn line from a weakly compact cardinal.

Aronszajn lines are tightly connected with *Aronszajn trees*: those trees in $\mathcal{T}_{\aleph_1}$ whose levels are countable. Each Aronszajn line is a linear extension of an Aronszajn tree order. Therefore, it is surprising that Todorcevic [26] showed that under $\mathsf{MA}_{\aleph_1}$, there cannot exist a universal Aronszajn tree. Thus, PFA, which implies $\mathsf{MA}_{\aleph_1}$, implies both the existence of a universal Aronszajn line and non-existence of a universal Aronszajn tree, despite the fact that there is a certain duality between these two. It is a longstanding open problem whether a universal Aronszajn tree can exist.

Question 1.1.3. *Is the existence of a universal $\aleph_1$-Aronszajn tree consistent?*

The methods from this thesis as such do not work with this problem, since in our set-up, when approximating a tree-embedding, it is essential to be able to separate a node from its image cofinally often with a small elementary submodel of some H_θ. If the levels of the tree are countable, they will be either completely contained, or completely disjoint from any model.

For the class of graphs, as mentioned above, the existence of a universal graph of size $\aleph_1$ alongside with the failure of the continuum hypothesis is consistent, as shown by Shelah in [24]. In Chapter 3 this result is generalized to the class of countably chromatic graphs of size $\aleph_1$. The proof is somewhat different than Shelah's proof from [24], for instance it avoids a preliminary forcing to add an almost disjoint family of subsets of ω_1, that is done in [24] in the case of (not necessarily countably chromatic) graphs.

Theorem 3.2.14. *The existence of a universal countably chromatic graph of size $\aleph_1$ along with the failure of continuum hypothesis is consistent.*

In Theorem 3.2.14, there is no need for large cardinal assumptions.

The proofs of Theorems 6.3.11, 5.3.12, 7.3.20 and 3.2.14 are forcing constructions, and in each of them, the preservation of cardinals is handled through a special use of side conditions - small models of set theory - as part of forcing conditions. In the case of the three first ones, the side conditions are models that are strong enough to reflect the second-order properties of the class in question (no cofinal branch). The existence of these is provided by the weakly compact cardinal.

Question 1.1.4. *Is the weakly compact needed in Theorems 6.3.11, 5.3.12 and 7.3.20?*

I would like to conjecture that the the answer is no. Instead of forcing using the reflective properties of the weakly compact cardinal and the weakly compact filter, one would try to force over L and use Devlin's principle $\diamondsuit^\#$ from [2] and the filter derived from it. This path towards solving Question 1.1.4 should be explored.

We end the section on universality problems with a difficult unprecise question.

Question 1.1.5. *Is there a way to render the methodology of Theorems 6.3.11, 5.3.12, 7.3.20 and 3.2.14 into a black box? Is there a forcing axiom hidden behind the proofs?*

As explained above, the proofs of Theorems 6.3.11, 5.3.12, 7.3.20 and 3.2.14 have the same structure, but differ in details. It would be nice to find, in the spirit of Mekler's result [17], a combinatorial condition P and a map $\mathcal{K} \mapsto \mathbb{P}_{\mathcal{K}}$ such that for any class $\mathcal{K}$ of objects of size κ satisfying the condition P, the poset $\mathbb{P}_{\mathcal{K}}$ is a poset that preserves κ

and forces that the class $\mathcal{K}$ has a universal object together with $\kappa^{<\kappa} > \kappa$. This is what is meant by "black box". The combinatorial condition P should be general enough to cover the classes $\mathcal{T}_{\aleph_1}$, $\mathcal{L}_{\aleph_1}$ and the class of countably chromatic graphs and other natural non-elementary classes, and simple enough to be easily verifiable. It would be even nicer if it turned out that something even better is around - these theorems could be glimpses of a hidden forcing axiom.

1.2 Wide Aronszajn trees

The purpose of this section is to describe the motivation behind the universality question in the class $\mathcal{T}_\kappa$. We call members of $\mathcal{T}_\kappa$ **wide κ-Aronszajn trees**. As said above, they are trees of size κ without branches of length κ. The interest in wide Aronszajn trees grew from abstract model theory; a completely different area of logic than where ordinary Aronszajn trees are traditionally studied. The starting point is *Scott analysis for countable models* that we briefly decribe now.

Definition 1.2.1. Let A and B be models in a same signature and let β be an ordinal. The **Ehrenfeucht-Fraïssé game $\mathsf{EF}^{\beta}(A, B)$** has two players and is played as follows: A position in the game is a pair (α, π), where $\alpha \leqslant \beta$ is an ordinal and π is a function from A to B. The starting position is $(\beta, \varnothing)$. At position (α, π), player I chooses an ordinal α' and a point $a \in A$ (or a point $b \in B$), and player II responds by finding a point $b \in B$ (or a point $a \in A$). The next position is $(\alpha', \pi \cup \{(a, b)\})$. The game ends if either $\alpha' \not< \alpha$ in which case player I loses, or $\pi \cup \{(a, b)\}$ is not a partial isomorphism, in which case player II loses.

The role of the ordinal β is to serve as a "game clock". Since the first player has to choose a descending sequence of ordinals, the game always results in a finite play, as there are no infinite descending sequences of ordinals. It follows that the game $\mathsf{EF}^{\beta}(A, B)$ is always determined. The Ehrenfeucht-Fraïssé game is a powerful tool to analyse countable models. In particular, it can be used to answer the question

> *How different are non-isomorphic models?*

Indeed, using the games EF^{β} it is possible to find a countable ordinal that can be seen as a parameter that measures the similarity of two countable models:

Theorem 1.2.2 (Scott [21]). *Two countable models A and B are non-isomorphic if and only if there is a countable ordinal $\beta < \omega_1$ such that player II has a winning stategy in $\mathsf{EF}^{\alpha}(A, B)$ for $\alpha \leqslant \beta$ and player I has a winning strategy in $\mathsf{EF}^{\gamma}(A, B)$ for $\gamma > \beta$.*

It is said that A and B are “isomorphic up to β" if player II has a winning strategy in $\mathsf{EF}^\beta(A,B)$. The games $\mathsf{EF}^\beta, \beta < \omega_1$ give rise to a “grading” of the isomorphism relation; to $\aleph_1$ many weaker relations that approximate $\cong$ in the class of countable models. Karp [10] gives a logical characterization for these relations in terms of the infinitary logic $\mathcal{L}_{\infty\omega}$ that extends the first-order logic by allowing infinitary conjunctions and disjunctions:

Theorem 1.2.3 (Karp [10]). *Let A and B be two structures in a countable relational signature. Player II has a winning strategy in $\mathsf{EF}^\beta(A,B)$ if and only if the models A and B are elementarily equivalent in the logic $\mathcal{L}_{\infty\omega}$ up to quantifier rank β.*

Thus if A and B are countable, then they are isomorphic if and only if they are elementarily equivalent in the logic $\mathcal{L}_{\infty\omega}$. In particular, for non-isomorphic models there is always an $\mathcal{L}_{\infty\omega}$-sentence (in fact, even an $\mathcal{L}_{\omega_1\omega}$-sentence) that separates them.

The game EF^β is also used to find invariants and classify countable models. This application exploits the game-logic correspondence to its full power. Scott [21] realized that it is possible to code winning strategies as $\mathcal{L}_{\omega_1\omega}$-sentences and use these to characterize countable models up to isomorphism:

Theorem 1.2.4 (Scott’s isomorphism theorem [21]). *For any countable model A there is an $\mathcal{L}_{\omega_1\omega}$-sentence ψ_A such that for any countable model B, B is isomorphic to A if and only if $B \models \psi_A$.*

The sentence ψ_A is called the **Scott sentence of A**.

It would be tempting to carry out the same analysis for uncountable models using the Ehrenfeucht-Fraïssé games corresponding to the logics $\mathcal{L}_{\lambda\kappa}$, but this does not work. Indeed, there are linear orders of size $\aleph_1$ that are elementarily equivalent even in the stronger logic $\mathcal{L}_{\infty\omega_1}$, and yet non-isomorphic. So, in general, it is impossible to separate non-isomorphic uncountable models using an $\mathcal{L}_{\infty\omega_1}$-sentence, or even, an Ehrenfeucht-Fraïssé game of length ω.

It was realised in the 90’s by members of the Helsinki school of logic that to develop Scott analysis for uncountable models, one has to play the Ehrenfeucht-Fraïssé game using trees as game clocks (see the survey [27]). The trees allow the game to have an arbitrarily long and yet countable length in the same way the ordinal clock β makes the game EF^β arbitrarily long but finite.

Definition 1.2.5. Let A and B be models in a same signature and let T be a tree. The game $\mathbf{EF}^{\boldsymbol{T}}(\boldsymbol{A},\boldsymbol{B})$ has two players and is played as follows: Positions in the game are pairs (C,π), where $C \subseteq T$ and π is a function from A to B. The starting position is $(\varnothing,\varnothing)$. At position (C,π), player I chooses a node t' and a point $a \in A$ (or a point

$b \in B$), and player II responds by choosing a point $b \in B$ (or a point $a \in A$). The next position is $(C \cup \{t'\}, \pi \cup \{(a, b)\})$. The game ends if either the node t' is not $<_T$-above every node in C, in which case player I loses, or else $\pi \cup \{(a, b)\}$ is not a partial isomorphism, in which case player II loses.

It can first be noted that if the tree T does not have an uncountable branch, then the game EF^T has countable but arbitrarily long length. Contrary to the game EF^β, the game EF^T might not be determined [6]. Moreover, if β is an ordinal and T_β is the tree of descending sequences of ordinals in β ordered by end-extension, then the games EF^β and EF^{T_β} are equivalent in the sense that any of the players has a winning strategy in one if and only if they have one also in the other. So the games EF^T with trees as clocks generalize the games EF^β with ordinals as clocks.

The game with trees as clocks is very effective in measuring the difference between two non-isomorphic models of size $\aleph_1$. To state the analogue of Theorem 1.2.2, we first need to introduce an operation, namely *Kurepa's σ-functor* from [15]. For a tree T, the tree σT consists of all the chains in T. The tree is ordered by end-extension. Let $S \leqslant T$ denote the fact that there exists a **weak embedding** from S to T; a function $f : S \to T$ that satisfies that if $s <_S t$, then $f(s) <_T f(t)$. Write $S \equiv T$ if both $S \leqslant T$ and $T \leqslant S$ hold and write $S < T$ if $S \leqslant T$ but $S \not\equiv T$. It can be shown that $T < \sigma T$ and that the σ-operation extends the successor operation on ordinals: $\sigma T_\alpha \equiv T_{\alpha+1}$ for any ordinal α. (In general, it does not hold that if $S < T$, then $\sigma S \leqslant T$.) If T has no uncountable branches, then neither does σT. If T is a wide $\aleph_1$-Aronszajn tree and the continuum hypothesis holds, then so is σT. However, in general σT might be much wider than T. The parameter that measures how different two non-isomorphic models of size $\aleph_1$ are, is a tree without uncountable branches:

Theorem 1.2.6 (Hyttinen, Väänänen [7]). *For models A and B of size $\aleph_1$, the following are equivalent:*

1. *$A \not\cong B$,*
2. *There are trees S and T without uncountable branches such that $S \leqslant T$ and*
    (a) *player II has a winning strategy in $\mathsf{EF}^T(A, B)$ but not in $\mathsf{EF}^{\sigma T}(A, B)$,*
    (b) *player I does not have a winning strategy in $\mathsf{EF}^S(A, B)$ but has one in $\mathsf{EF}^{\sigma S}(A, B)$.*

This theorem is stated in two parts since the Ehrenfeucht-Fraïssé games with trees as games clocks are not necessarily determined [6].

While Theorem 1.2.6 answers the question how to separate non-isomorphic uncountable models, no complete analogue for Scott's characterization of model up to isomorphism by its Scott sentence (Theorem 1.2.4) has been found. The work is not finished, only partial results exist thus far. In the case of countable models, the logic that corresponds the game EF^{β} is the logic $\mathcal{L}_{\infty\omega}$. It is an empirical fact that Ehrenfeucht-Fraïssé type of games always correspond to a logic. In some sense this is the case for the games EF^{T} too. There exists a "tree logic" $\mathcal{L}_{\mathcal{T}}$ where trees without uncountable branches work as quantifier ranks. The logic has a generative set of formulas, i.e. the set of formulas is obtained from atomic formulas by closing under a certain set of operations. There is a drawback: the semantics is defined through a variation of a satisfaction game where a tree is used as a clock, which results in a logic that is not necessarily closed under negation, due to the fact that the satisfaction game might not be determined. However, against certain trees the game is always determined, and for some uncountable models A there does exist a tree T such that for any B, player II has a winning strategy in $\mathsf{EF}^{T}(A,B)$ if and only if $A \cong B$. In this case there also is a Scott sentence ψ_A for A in the tree logic $\mathcal{L}_{\mathcal{T}}$. See the survey [27] or [11] and [6] for more.

The main results of this thesis are that the existence of a universal wide κ-Aronszajn tree is consistent for $\kappa = \aleph_1$ or κ a double successor cardinal (modulo consistency of a weakly compact cardinal). The reason why this scenario among the structure of these "clock trees" is interesting from the point of view of classifying uncountable structures is the following simple observation: if player II has a winning strategy in the game $\mathsf{EF}^{T}(A,B)$ and $S \leqslant T$, then she has a winning strategy also in the game $\mathsf{EF}^{S}(A,B)$. Thus, in order to verify the similarity of A and B up to a set of potentially incomparable trees T_i, $i \in I$, it suffices to verify the similarity of A and B up to a tree that is above any tree T_i. Thus, the existence of a universal wide $\aleph_1$-Aronszajn tree acts as a "master tree" for all the clock trees of size $\aleph_1$.

Chapter 2

Preliminaries

The consistency proofs of the existence of a universal object in this thesis have a similar architecture. The first poset introduces a generic object $\dot{a}$ to the class $\mathcal{K}$ in question, and then an iteration is performed in order to create embeddings from other members of $\mathcal{K}$ into $\dot{a}$. Preservation of the relevant cardinal is guaranteed by strong properness with respect to a relevant class of models (See Definition 2.3.1). Adding *side conditions* help crafting posets that are proper or even strongly proper. These are small elementary submodels of some H_θ that are added to a forcing poset to control the intended generic object. Usually, when forcing with side conditions, each condition has two parts: the working part and the side condition part. The working part is intended to build the desired generic object, in this thesis it will be an approximation of an embedding from an object in $\mathcal{K}$ to $\dot{a}$. The side condition part is there to take care of preservation of cardinals. It is a small set of models with respect to which we intend to show strong properness. Here, for example, it is required that each model in this small set is closed for the embedding approximation, i.e. the working part, among other things.

In the proof of chapters 5, 6 and 7, we will fix a weakly compact cardinal κ and use the normal κ-complete weakly compact filter $\mathcal{F}_{\text{wc}}$ (see Definition 2.2.1 and Lemma 2.2.2) to guide the side conditions in the iteration. At each step in the iteration, there will be a set in $\mathcal{F}_{\text{wc}}$ that indexes models used that that stage. The structure of these sets and models is described in Chapter 4. The construction is very flexible and generalizes to any other normal filter $\mathcal{F}$.

The notation is standard and follows [8]. For a set X and a cardinal κ, $\mathscr{P}_\kappa(X)$ is the set of subsets of X of size $< \kappa$. The set H_κ is the set of sets of hereditary cardinality $< \kappa$. If κ is a regular cardinal, then $(H_\kappa, \in)$ is a model of ZFC^-, i.e. all the axioms of set theory except potentially the power set axiom. By Card we denote the class of infinite cardinals. For a forcing poset $\mathbb{P}$ and $p, q \in \mathbb{P}$, we write $q \leqslant p$ to denote that the condition q is stronger than p.

2.1 Trees

A **tree** is a partial order $(T, <_T)$ such that the set of predecessors of any node is well-ordered by $<_T$. A **wide $\boldsymbol{\kappa}$-tree** is a tree of size κ and a tree is **normal** if the meet of two nodes is always well-defined and unique. The meet is denoted by $\boldsymbol{s \wedge t}$ and it is the maximal node that is below or equal to both s and t. For simplicity, we consider only normal trees. For a node $t \in T$, the **height of $\boldsymbol{t}$** is the order-type of the set of predecessors of t ordered by $<_T$. The **$\boldsymbol{\alpha}$-th level** of a tree T is the set of nodes of height α.

There are two possible choices of morphism among trees.

Definition 2.1.1. Let $f : S \to T$ be a function between two trees. Then f is a

1. **weak embedding** if for all $s, t \in S$,
$$s <_S t \implies f(s) <_T f(t).$$
2. **strong embedding** if f is injective, level-preserving and for all $s, t \in S$,
$$s <_S t \iff f(s) <_T f(t).$$

The following properties are well known. See for instance [26].

Lemma 2.1.2. *Let S and T be trees.*

1. *If there is a weak embedding $f : S \to T$, then there is a level-preserving one.*
2. *A level-preserving map $f : S \to T$ is a strong embedding if and only if it satisfies for all $s, t \in S$ that*
$$f(s \wedge t) = f(s) \wedge f(t).$$

Moreover, if the domain of a partial level-preserving weak embedding $f : S \to T$ is closed under meets, it is a strong embedding if and only if it satisfies $f(s \wedge t) = f(s) \wedge f(t)$ for all $smt \in \mathsf{dom}(f)$.

2.2 Weakly compact cardinals

A formula φ is a **$\boldsymbol{\Pi^1_1}$-formula over a set $\boldsymbol{M}$** if φ has form $\forall X \psi(X, \bar{A}, \bar{a})$, where $\bar{A} = (A_1, \ldots, A_n)$ is a tuple of subsets of M and $\bar{a} = (a_1, \ldots a_m)$ is a tuple of elements of M, and X is a second-order variable. Second-order variables are interpreted as unary predicates.

Definition 2.2.1. An uncountable cardinal κ is **weakly compact** if for every Π^1_1-formula φ and $A \subseteq V_\kappa$ such that

$$V_\kappa \models \varphi(A),$$

there is $\alpha < \kappa$ such that

$$V_\alpha \models \varphi(A \cap V_\alpha).$$

If κ is weakly compact, then the sets of the form

$$X_{A,\varphi} = \{\alpha < \kappa : \text{if } V_\kappa \models \varphi \text{ then } V_\alpha \models \varphi(A \cap V_\alpha)\}$$

generate a filter on κ. This filter is called the **weakly compact filter** and denoted by $\boldsymbol{\mathcal{F}_{\mathrm{wc}}}$. We say that a filter on κ is **normal** if it is closed under *diagonal intersections* of length κ:

$$\Delta_{i<\kappa} A_i = \{\beta < \kappa : \beta \in \bigcap_{\alpha<\beta} A_\alpha\}.$$

Lemma 2.2.2. *The weakly compact filter $\mathcal{F}_{\mathrm{wc}}$ is normal and extends the club-filter on κ.*

Proof. See Chapter 1 section 6 from [9]. □

The following observations are used in Chapters 5, 6 and 7. They can be proved by applying Lemma 13.12 from [8] to the recursive definition of the forcing relation.

Observation 2.2.3. *Let κ be a cardinal. If φ is a Π^1_1-formula over V_κ and $\mathbb{P} \subseteq V_\kappa$ is a poset, then so is the formula*

$$``p \Vdash_{\mathbb{P}} \varphi".$$

Furthermore, for every $\mathbb{P}$-name there is an equivalent $\mathbb{P}$-name that is a subset of V_κ.

Observation 2.2.4. *If $\mathbb{P}$ is a poset such that $\mathbb{P} \subseteq V_\kappa$, then the formula*

$$``p \Vdash_{\mathbb{P}} \dot{T} \textit{ is a wide } \kappa\textit{-Aronszajn tree}"$$

is a Π^1_1-formula over V_κ.

2.3 Strong properness

Strong properness is a strengthening of properness that has a convenient combinatorial formulation in terms of *residue conditions* that is potentially easier to verify than properness. However, not many posets are strongly proper. For instance σ-closed posets are not, so there is a trade-off for the convenience. But they are convenient to work with, especially when forcing Aronszajn trees, since strongly proper posets cannot add long branches to trees (see Lemma 2.3.7). The notion and the basic properties of strongly proper posets are due to Mitchell [18].

Definition 2.3.1. Let $\mathbb{P}$ be a poset and let M be a set.

1. Let $p \in \mathbb{P}$. A condition $r \in \mathbb{P} \cap M$ is a **residue of p into M** if every $w \in \mathbb{P} \cap M$ that extends r is compatible with p.
2. A condition $p \in \mathbb{P}$ is **strongly $(\mathbb{P}, M)$-generic** if every $q \leqslant p$ has a residue into M.
3. The poset $\mathbb{P}$ is **strongly proper with respect to M** if for every $p \in \mathbb{P} \cap M$ there is $q \leqslant p$ that is strongly $(\mathbb{P}, M)$-generic.

Furthermore, we say that $\mathbb{P}$ is **κ-strongly proper** if it is strongly proper with respect to stationarily many $M \in [H_\theta]^{<\kappa}$ for any arbitrarily large regular θ, and $\mathbb{P}$ is simply **strongly proper** if it is $\aleph_1$-strongly proper.

A set M is called **suitable** for a poset $\mathbb{P}$ if $(M, \in)$ is an elementary submodel of some $(H(\theta), \in)$, where θ is a large enough regular cardinal, and $\mathbb{P} \in M$.

A proof for the following lemma can be found in [5]:

Lemma 2.3.2. *Let $\mathbb{P}$ be a poset and let M be suitable for $\mathbb{P}$. The following are equivalent for a condition $p \in \mathbb{P}$:*

1. *p is strongly $(\mathbb{P}, M)$-generic,*
2. *the set of conditions that have a residue into M is dense below p,*
3. *$p \Vdash$ "$\check{G} \cap M$ is V-generic on $\mathbb{P} \cap M$".*

Lemma 2.3.2 says that if $\mathbb{P}$ is strongly proper with respect to M, then $\mathbb{P} \cap M$ is a complete subposet of $\mathbb{P}$ "modulo a strongly $(\mathbb{P}, M)$-generic condition". Those V-generic filters on $\mathbb{P}$ that contain a strongly $(\mathbb{P}, M)$-generic condition restrict to a generic on $\mathbb{P} \cap M$. Conversely, small generics extend on larger posets, again modulo a strongly generic condition:

Lemma 2.3.3. *Assume that M is a suitable model for $\mathbb{P}$ and p is strongly $(\mathbb{P}, M)$-generic. Then every V-generic filter on $\mathbb{P} \cap M$ that contains a residue of p extends to a V-generic on $\mathbb{P}$ that contains p.*

If $G \subseteq \mathbb{P} \cap M$ is a V-generic filter, we define

$$\mathbb{P}/G := \{p \in \mathbb{P} : p \text{ is compatible with every } w \in G\}.$$

Then, a condition $r \in \mathbb{P} \cap M$ is a residue of p into M if and only if $r \Vdash$ "$p \in \mathbb{P}/\check{G}_{\mathbb{P}\cap M}$". It follows from the above lemmas that posets that have strongly generic conditions factor into two, modulo these conditions. Specifically, if p is strongly $(\mathbb{P}, M)$-generic, then any residue r of p forces that $\mathbb{P}/p = \{q : q \leqslant p\}$ is forcing equivalent to the two-step iteration

$$(\mathbb{P} \cap M) * (\mathbb{P}/p)/\check{G}_{\mathbb{P}\cap M}.$$

It follows from Lemma 2.3.2 that strong properness is a strengthening of properness[1]. Thus, preservation of cardinals can be expected. The following fact is standard and a proof can be found for instance in [8].

Lemma 2.3.4. *Let κ be a regular cardinal. Any κ-strongly proper poset preserves κ.*

In particular, strongly proper forcing preserves $\aleph_1$. Contrary to some proper posets, strongly proper posets always add lots of reals. The following lemma is also well known.

Lemma 2.3.5. *If $\mathbb{P}$ is κ-strongly proper, then it adds Cohen subsets to stationarily many $\alpha < \kappa$.*

In particular, Lemma 2.3.5 shows that κ-strongly proper posets can never be $< \kappa$-closed. However, it can have κ-cc, but this is not guaranteed. The following lemma is standard and a proof can be found in [23].

Lemma 2.3.6. *A poset $\mathbb{P}$ has κ-cc if an only if the top condition $1_{\mathbb{P}}$ is $(\mathbb{P}, M)$-generic with respect to club many $M \in [H_\theta]^{<\kappa}$ for any large enough regular θ.*

What is particularly nice about strongly proper posets is that they do not add branches to trees:

---

[1] A condition is **$(\mathbb{P}, M)$-generic** if it forces that $\check{G} \cap M$ is an M-generic filter on $\mathbb{P} \cap M$. A poset is **proper** if any condition in $\mathbb{P} \cap M$ has a $(\mathbb{P}, M)$-generic strengthening.

Lemma 2.3.7. *A poset that is κ-strongly proper does not add branches of length κ to trees in V.*

For a proof of Lemma 2.3.7, see for instance [20].

Let $\kappa \leqslant \lambda$ be regular cardinals. The Levy collapse poset $\mathsf{Col}(\kappa, < \lambda)$ is the poset of partial functions $p : \lambda \times \kappa \to \lambda$ of size $< \kappa$ that satisfy $p(\alpha, \beta) < \alpha$, ordered by inverse inclusion.

Example 2.3.8. Let $\kappa \leqslant \lambda$ be regular cardinals and let M be a model of ZFC^- of size $< \lambda$ that satisfies $\lambda \cap M \in \mathsf{Ord}$ and contains κ. Then the Levy Collapse $\mathsf{Col}(\kappa, < \lambda)$ is strongly proper with respect to $M$with the pointwise intersection $p \cap M$ being a residue of p into M for any condition $p \in \mathsf{Col}(\kappa, < \lambda)$.

For more about the general theory of strong properness see for example [5].

Chapter 3

Countably chromatic graphs

A graph is **countably chromatic** if there exists a coloring function on the set of vertices into $\mathbb{N}$ that assigns different colors to adjacent vertices. In this section we consider the universality problem in the class of countably chromatic graphs. The consistency proof of the existence of universal object in this class is significantly simpler than the later proofs, for instance it involves side conditions only implicitly. Therefore, this section serves as a kind of a “warm-up” for the next sections.

More specifically, we consider the universality problem in the class of graphs together with a fixed coloring; the objects are triples $G = (G, E, c)$, where (G, E) is an undirected and loopless graph with $G \subseteq \omega_1$ and $c : G \to \omega$ is a function such that for all vertices $a, b \in G$,

$$c(a) = c(b) \implies \neg E(a, b).$$

We refer to the function c by **coloring** of G. An **embedding** between colored graphs (G, E_G, c_G) and (H, E_H, c_H) is an injective function $f : G \to H$ preserving the edge relation and the coloring:

1. $E_G(a, b)$ if and only if $E_H(f(a), f(b))$,
2. $c_G(a) = c_H(f(a))$.

We refer to this class of objects and embeddings by **countably chromatic graphs of size $\aleph_1$**. The main theorem of this chapter is the following.

Theorem 3.2.14. *It is consistent that the class of countably chromatic graphs of size* $\aleph_1$ *has a universal object and* $2^\omega = \aleph_2$.

The history of the universal countably chromatic graph of size $\aleph_1$ begins with the Rado graph, or random graph, which is a countable graph that is universal for the class of countable graphs. The first natural question was whether there exists an analogue of such graph in size $\aleph_1$. It turned out that the existence of a graph of size $\aleph_1$ that is both universal and *strongly homogeneous*[1] is in fact equivalent to the continuum hypothesis (See Theorem 4.7 on page 476 in [22]). Shelah [24] then showed that the existence of a graph of size $\aleph_1$ that is universal but not necessarily strongly homogeneous is consistent with the failure of the continuum hypothesis. Also the non-existence is: after adding $\aleph_2$ many Cohen reals, there is no universal graph of size $\aleph_1$ and the continuum hypothesis fails. For a proof, see appendix of [14].

The purpose of this section is to show that this picture generalizes to countably chromatic graphs. The continuum hypothesis implies the existence of a universal countably chromatic graph of size $\aleph_1$. The proof of this fact is similar as in the case of graphs, despite the fact the class of countably chromatic is not elementary (first-order definable). The trick is to move to a many-sorted signature with one sort for each color. By the continuum hypothesis, there is a saturated model of size $\aleph_1$ for this many-sorted theory, which is universal. (See [25] for why the continuum hypothesis implies the existence of a saturated model.)

Likewise, it holds also that after adding $\aleph_2$ many Cohen reals there is no universal countably chromatic graph of size $\aleph_1$ and the proof is a straightforward modification of the proof in the appendix of [14].

The main theorem (Theorem 3.2.14) is proved by forcing. We start with a model of GCH and perform a finite support iteration of length ω_2 of strongly proper ccc posets. The structure of the construction is inspired by [1] and [13] and the definition of the poset is inspired by [24]. The proof here is different from Shelah's proof, in presentation and also in that, for instance, he makes use of a preparatory forcing to add a family of almost disjoint sets, that is not necessary here.

The same proof works for higher successor cardinals: for any cardinal κ, there is a poset that forces the existence of a graph of size κ^+ of chromatic number κ that contains an isomorphic copy of each such graph, together with $2^\kappa > \kappa^+$.

3.1 Poset

We assume GCH in the ground model. The poset $\mathbb{P}$ will be a direct limit of a finite support iteration $(\mathbb{P}_\delta : \delta \leqslant \omega_2)$, where

---

[1] every small partial elementary map lifts to an automorphism

1. $\mathbb{P}_1$ builds a generic graph structure $\dot{\mathcal{G}}$ on ω_1 together with a generically chosen coloring function,
2. the tail of the iteration creates embeddings
$$\dot{f}_\delta : \dot{H}_\delta \to \dot{\mathcal{G}},$$
where $\dot{H}_\delta$ is a $\mathbb{P}_\delta$-name for a countably chromatic graph on ω_1 chosen by a suitable bookkeeping function, for every $\delta < \omega_2$.

In the poset, carefully chosen small elementary submodels of some H_θ will serve as auxiliary tool to control the domain and the range of the graph embedding approximations. The iteration will be defined in such a way that at each stage $\delta < \omega_2$, we will fix a sequence of countable models $(M^\delta_\alpha : \alpha \in \mathcal{E}_\delta)$ indexed by a club $\mathcal{E}_\delta \subseteq \omega_1$, and it will be made sure that the top condition $1_\mathbb{P} \in \mathbb{P}_\delta$ is strongly $(\mathbb{P}_\delta, M^\delta_\alpha)$-generic, for every $\alpha \in \mathcal{E}_\delta$. The role of these models is similar to side conditions, with the exception that they do not explicitly appear in the poset. It will follow that the final poset $\mathbb{P}_{\omega_2}$ has ccc.

We begin with the first step, which is to force a generic countably chromatic graph of size $\aleph_1$. This is done with finite conditions, with a poset equivalent to the poset $\mathsf{Add}(\omega, \omega_1)$ that adds $\aleph_1$ many Cohen reals. We assume that GCH holds in the ground model.

Definition 3.1.1 ($\mathbb{P}_1$)**.** The poset $\mathbb{P}_1$ consists of triples
$$p = (u^p, E^p, c^p),$$
where

1. $u^p \subseteq \omega_1$ is a finite set,
2. $E^p : [u^p]^2 \to 2$ is a partial function,
3. $c^p : u^p \to \omega$ is a function such that if $c^p(s) = c^p(t)$ and $\{s,t\} \in \mathsf{dom}(E^p)$, then $E^p(\{s,t\}) = 0$.

The ordering is by $q \leqslant p$ iff $u^q \supseteq u^p$, $E^q \supseteq E^p$ and $c^q \supseteq c^p$.

A generic filter $G \subseteq \mathbb{P}_1$ gives a graph on ω_1 by letting
$$E^G(s,t) :\iff \exists p \in G\ E^p(\{s,t\}) = 1,$$
and the countable chromaticity of this graph is witnessed by the coloring
$$c^G := \bigcup_{p \in G} c^p.$$

Notation 3.1.2. Let $\dot{\mathcal{G}}$ be a $\mathbb{P}_1$-name for the graph with a coloring $(\omega_1, E^{\check{G}}, c^{\check{G}})$.

The poset $\mathbb{P}_1$ is equivalent to the poset that adds a subset of ω_1 with finite conditions, or equivalently to the poset $\mathsf{Add}(\omega, \omega_1)$ that adds ω_1 many Cohen reals. The goal is to define a poset $\mathbb{P}_{\omega_2}$ which creates an embedding

$$\dot{f}_\delta : \dot{H}_\delta \to \dot{\mathcal{G}}$$

for each $\delta < \omega_2$, where $\dot{H}_\delta$ is a countably chromatic graph on ω_1 chosen by a suitable bookkeeping function. The first poset will be $\mathbb{P}_1$. Being countably chromatic is preserved by any forcing extension, so when forcing the embeddings, we only need to worry about the preservation of $\aleph_1$.

For each $\delta < \omega_2$, we define models $(M^\delta_\alpha : \alpha < \omega_1)$ which will implicitly play the role of side-conditions, even if they will not explicitly appear as part of the conditions in $\mathbb{P}_\delta$. For each poset $\mathbb{P}_\delta$, there will be a club $\mathcal{E}_\delta \subseteq \omega_1$ such that $\mathbb{P}_\delta$ is strongly proper with respect to each model M^δ_α for $\alpha \in \mathcal{E}_\delta$. These clubs $\mathcal{E}_\delta$ will get "faster" in the sense that whenever $\alpha \in \mathcal{E}_\delta$ and $\gamma \in \delta \cap M^\delta_\alpha$, then α is a limit point of $\mathcal{E}_\gamma$.

Notation 3.1.3. We fix a regular cardinal $\theta > \aleph_2$ and a well-ordering $<_\theta$ of $H(\theta)$. We also fix a bookkeeping function

$$\omega_2 \to H(\omega_2), \quad \gamma \mapsto \dot{H}_\gamma$$

such that whenever the poset $\mathbb{P}_\gamma$ is defined, then $\dot{H}_\gamma$ is a $\mathbb{P}_\gamma$-name for a countably chromatic graph on ω_1. We assume that for every $\gamma < \omega_2$ and for every $\mathbb{P}_\gamma$-name for a countably chromatic graph on ω_1 there is a name for an isomorphic copy of it in the range of the bookkeeping function. We assume that the bookkeeping function is a minimal such function with respect to $<_\theta$.

The posets $\mathbb{P}_\delta$ and clubs $\mathcal{E}_\delta \subseteq \omega_1$ are defined by recursion on $\delta < \omega_2$. The first poset $\mathbb{P}_0$ will be the trivial poset $\{\varnothing\}$. The poset $\mathbb{P}_1$ was already defined in 3.1.1.

For a set $X \subseteq \omega_1$, the set $\lim X$ is the set of limit points of X.

Definition 3.1.4. Let $\delta < \kappa^+$. Whenever the poset $\mathbb{P}_\delta$ and the sets $\bar{\mathcal{E}}_\delta := (\mathcal{E}_\gamma)_{\gamma<\delta}$ are defined, we let:

1. For every $\alpha \leqslant \kappa$, let

$$M^\delta_\alpha := \text{Skolem hull of } \alpha \cup \{\delta, \mathbb{P}_\delta, \bar{\mathcal{E}}_\delta\} \text{ in } (H(\theta), \in, <_\theta),$$

2. Let

$$\mathcal{E}_\delta := \{\alpha < \kappa : \kappa \cap M^\delta_\alpha = \alpha \text{ and } \alpha \in \bigcap_{\gamma \in \delta \cap M^\delta_\alpha} \lim \mathcal{E}_\gamma\}.$$

Moreover, for $\alpha < \omega_1$, let

$$\begin{aligned}\rho_\delta(\alpha) &:= \text{ the least element in } \mathcal{E}_\delta \text{ strictly above } \alpha,\\ \lambda_\delta(\alpha) &:= \text{ the least limit point of } \mathcal{E}_\delta \text{ strictly above } \rho_\delta(\alpha).\end{aligned}$$

Then each $\mathcal{E}_\delta$ is a club, by normality of the club filter. For every $\alpha < \kappa$, we have

$$\alpha < \rho_\delta(\alpha) < \lambda_\delta(\alpha).$$

When mapping the δ-th graph $\dot{H}_\delta$ on ω_1 into $\dot{\mathcal{G}}$, we will use finite partial functions $f : \omega_1 \to \omega$ such that each $\alpha \in \mathsf{dom}(f)$ is mapped into the interval $f(\alpha) \in [\rho_\delta(\alpha), \lambda_\delta(\alpha))$. The ordinal $\rho_\delta(\alpha)$ is used as a "rank" that separates each vertex α in $\dot{H}_\delta$ from its image $f(\alpha)$.

For a technical reason that appears in the proof of strong properness, we need to label each node in $\dot{\mathcal{G}}$ with a countable ordinal. This label will be used to read off the $\rho_\delta(\alpha)$ from the preimage α of a node $\beta \in \dot{\mathcal{G}}$.

Notation 3.1.5. We fix a function $\ell : \omega_1 \to \omega_1$ such that

1. $\ell(\alpha) < \alpha$ for every non-zero $\alpha < \omega_1$,
2. for every $\alpha < \omega_1$ there are unboundedly many $\beta > \alpha$ with $\ell(\beta) = \alpha$.

We call $\ell(\alpha)$ the **label** of α.

We are ready to define the embedding posets. The definition is recursive. In addition to the separation of each vertex α in the δ-th graph $\dot{H}_\delta$ from its image in $\dot{\mathcal{G}}$ using the ordinal $\rho_\delta(\alpha)$, we need to be able to read off $\rho_\delta(\alpha)$ from the image of α. We will make sure to embed α to some β whose label $\ell(\beta)$ is the ordinal $\rho_\delta(\alpha)$.

The clubs $\mathcal{E}_\gamma$, $\gamma < \delta$ will be used in the definition of the poset $\mathbb{P}_\delta$, and the club $\mathcal{E}_\delta$ is defined once the poset $\mathbb{P}_\delta$ is defined.

Definition 3.1.6 ($\mathbb{P}_\delta$). Let $\delta \leqslant \omega_2$. Assume that $\mathbb{P}_\gamma$ and $\mathcal{E}_\gamma$ are defined for $\gamma < \delta$. The conditions in $\mathbb{P}_\delta$ are functions $p : \delta \to H(\omega_2)$ satisfying the following:

1. $p(0) = (u^p, E^p, c^p) \in \mathbb{P}_1$,

2. For non-zero $\gamma < \delta$, $p(\gamma)$ is a pair
$$p(\gamma) = (f^p_\gamma, S^p_\gamma),$$
where
    (a) $f^p_\gamma : \omega_1 \to u^p$ is a finite partial injective function,
    (b) $f^p_\gamma(\alpha) \in [\rho_\gamma(\alpha), \lambda_\gamma(\alpha))$ for all $\alpha \in \mathsf{dom}(f^p_\gamma)$,
    (c) the label of $f^p_\gamma(\alpha)$ satisfies $\ell(f^p_\gamma(\alpha)) = \rho_\gamma(\alpha)$,
    (d) $p \restriction \gamma$ decides the $\dot{H}_\gamma$-edges and colors in $\mathsf{dom}(f^p_\gamma)$,
    (e) $p \restriction \gamma \Vdash$ "$f^p_\gamma : \dot{H}_\gamma \to \dot{\mathcal{G}}$ is edge- and color-preserving",
    (f) $S^p_\gamma \subseteq \omega_1$ is a finite set and $\mathsf{ran}(f^p_\gamma) \cap S^p_\gamma = \varnothing$,
3. the **support** $\mathsf{sp}(p) := \{\gamma < \delta : p(\gamma) \neq (\varnothing, \varnothing)\}$ is finite.

The ordering is defined by $q \leqslant p$ if:

1. $q(0) \leqslant p(0)$,
2. $f^q_\gamma \supseteq f^p_\gamma$ and $S^q_\gamma \supseteq S^p_\gamma$ for every non-zero $\gamma < \delta$.

The posets $\mathbb{P}_\delta$ as well as the auxiliary sets $\mathcal{E}_\delta$, $\delta \leqslant \omega_2$, have now been defined. We will show first a density claim about the embeddings, and then that each poset $\mathbb{P}_\delta$ has ccc. This will be done by showing that every condition $p \in \mathbb{P}_\delta$ is strongly $(\mathbb{P}_\delta, M^\delta_\alpha)$-generic, for $\delta < \omega_2$ and $\alpha \in \mathcal{E}_\delta$. It will follow that each $\mathbb{P}_\delta$ is strongly proper and has ccc, and thus $\mathbb{P}_{\omega_2}$ will have ccc, being the direct limit of ccc posets.

The following is clear from definitions:

Lemma 3.1.7. *If $\gamma \leqslant \delta \leqslant \omega_2$, $p \in \mathbb{P}_\delta$ and $q \in \mathbb{P}_\gamma$ extends $p \restriction \delta$, then the concatenation*
$$q^\frown p \restriction [\gamma, \delta)$$
is a condition in $\mathbb{P}_\delta$ that extends p.

In particular, each poset $\mathbb{P}_\gamma$ embeds completely into $\mathbb{P}_\delta$, for $\gamma \leqslant \delta \leqslant \omega_2$.

Next we show that in $V^{\mathbb{P}_{\omega_2}}$, the graph $\dot{\mathcal{G}}$ contains an isomorphic copy of each graph $\dot{H}_\delta$. In other words, we show that if $G \subseteq \mathbb{P}_{\omega_2}$ is a generic, then for each $\delta < \omega_2$, the map $f^G_\delta := \bigcup_{p \in G} f^p_\delta$ satisfies $\mathsf{dom}(f^G_\delta) = H^G_\delta$. It suffices to show that for every $\delta < \omega_2$ and $\alpha < \omega_1$ the set $\{p \in \mathbb{P}_{\omega_2} : \alpha \in \mathsf{dom}(f^p_\delta)\}$ is dense.

The following is immediate:

Lemma 3.1.8. *Let $G \subseteq \mathbb{P}_{\omega_2}$ be a generic filter.*

1. *For every $\delta < \omega_2$, the function*

$$f_\delta^G := \bigcup_{p \in G} f_\delta^p : \dot{H}_\delta^G \to \dot{\mathcal{G}}^G$$

   is edge- and color-preserving.

2. *The function*

$$c^G := \bigcup_{p \in G} c^p : \dot{\mathcal{G}}^G \to \omega$$

   is a coloring of $\dot{\mathcal{G}}^G$, witnessing that $\dot{\mathcal{G}}^G$ is countably chromatic.

We show that the generic embeddings are defined on the whole domain of $\dot{H}_\delta$ for each $\delta < \omega_2$.

Lemma 3.1.9. *For every $p \in \mathbb{P}_{\omega_2}$, $\delta < \omega_2$ and every $\alpha < \omega_1$ there is $q \leqslant p$ such that $\alpha \in \mathsf{dom}(f_\gamma^q)$.*

Proof. Up to extending $p \restriction \delta$, we may assume that it decides the $\dot{H}_\delta$-edges and colors in the set $\mathsf{dom}(f_\delta^p) \cup \{\alpha\}$. The interval $[\rho_\gamma(\alpha), \lambda_\gamma(\alpha))$ is an infinite subset of $\mathcal{G}$ and the set $u^p \cup S_\delta^p$ is finite, so we may find a node $\beta \in [\rho_\gamma(\alpha), \lambda_\gamma(\alpha)) - (u^p \cup S_\delta^p)$ and extend $p(0)$ by assigning the color of α to β and connecting it to those nodes $f_\delta^p(\alpha') \in u^p$ such that $p \restriction \delta \Vdash$ ``$E_{\dot{H}_\delta}(\alpha, \alpha')''$ and disconnect it from all other nodes in u^p. Define q such that $q(0)$ is the resulting extension of $p(0)$ and

$$q(\gamma) = \begin{cases} (f_\gamma^p \cup \{(\alpha, \beta)\}, S_\gamma^p), & \text{if } \gamma = \delta, \\ p(\gamma), & \text{otherwise.} \end{cases}$$

This q is a condition, extends p and has α in the domain of f_δ^q. $\square$

3.2 Strong properness and ccc

It remains to show that ω_1 is preserved and that $2^\omega = \omega_2$ in $V^{\mathbb{P}_{\omega_2}}$. This is done by showing that each $\mathbb{P}_\delta$ is strongly proper with respect to the models

$$\{M_\alpha^\delta : \alpha \in \mathcal{E}_\delta\}.$$

In fact, we will show that the top condition $1_{\mathbb{P}_\delta}$ is strongly $(\mathbb{P}_\delta, M^\delta_\alpha)$-generic, for each $\alpha \in \mathcal{E}_\delta$. This implies that each $\mathbb{P}_\delta$ has ccc (see [23]). In order to show that $1_{\mathbb{P}_\delta}$ is strongly $(\mathbb{P}_\delta, M^\delta_\alpha)$-generic, it suffices to show that for densely many conditions p, the pointwise intersection of p with M^δ_α, its *trace*, is a residue of p into M^δ_α.

Definition 3.2.1. Let $\delta < \omega_2$ and $\alpha \in \mathcal{E}_\delta$. The **trace** of a condition $p \in \mathbb{P}_\delta$ into M^δ_α is defined to be

$$[p]^\delta_\alpha : \delta \to M^\delta_\alpha, \quad p(\gamma) := \begin{cases} (u^p \cap M^\delta_\alpha, E^p \cap M^\delta_\alpha, c^p \cap M^\delta_\alpha) & \text{if } \gamma = 0, \\ (f^p_\gamma \cap M^\delta_\alpha, S^p_\gamma \cap M^\delta_\alpha), & \text{if } \gamma \in \delta \cap M^\delta_\alpha - \{0\}, \\ \varnothing, & \text{if } \gamma \notin \delta \cap M^\delta_\alpha. \end{cases}$$

The trace might not even be a condition. It will be one whenever p is *super-nice*, in which case it will be a residue of p as well.

Remark 3.2.2. It follows from the definition of the poset that if $\delta < \omega_2, \alpha \in \mathcal{E}_\delta, p \in \mathbb{P}_\delta$ and $\gamma \in \delta \cap M^\delta_\alpha$, then the model M^δ_α is closed for the function f^p_γ: if $\alpha' \in \mathsf{dom}(f^p_\gamma) \cap M^\delta_\alpha$, then $\alpha' < \alpha$. Since $\gamma \in M^\delta_\alpha$, the ordinal α must be a limit point of $\mathcal{E}_\gamma$. Thus $\lambda_\gamma(\alpha') \leqslant \alpha$. Now

$$f^p_\gamma(\alpha') \in [\rho_\gamma(\alpha'), \lambda_\gamma(\alpha')) \subseteq \alpha \subseteq M^\delta_\alpha.$$

So $f^p_\gamma(\alpha') \in M^\delta_\alpha$.

Remark 3.2.3. The sets S^p_γ are used as follows. Suppose that $p \in \mathbb{P}_\delta$ is a condition, $\alpha \in \mathcal{E}_\delta$, and we are showing that the trace $[p]^\delta_\alpha$ is a residue of p into M. Let $\gamma < \delta$, $\gamma \in M^\delta_\alpha$. Suppose for simplicity that the graph $\dot{H}_\gamma$ is in V. By Remark 3.2.2 above the model M^δ_α is closed under the function f^p_γ.

- Suppose that $\beta_1 \in \omega_1 \cap M^\delta_\alpha$, $\beta_2 \in \omega_1 - M^\delta_\alpha$ and f^p_γ maps some node $\alpha_2 \in M$ to β_2, but no node to β_1.
- Suppose that $w \in \mathbb{P}_\delta \cap M^\delta_\alpha$ extends $[p]^\delta_\alpha$ by sending some α_2 to β_2.
- If there is an edge $E_{H_\gamma}(\alpha_1, \alpha_2)$ but $p \Vdash \neg E^{\check{G}}(\beta_1, \beta_2)$, or vice versa, then w cannot be compatible with p.
- We put into S^p_γ every node in $M^\delta_\alpha - \mathsf{ran}(f^p_\gamma)$ of which p decides that it is connected or disconnected from a vertex outside of M. There are finitely many of these and they all are in u^p.
- In particular, the vertex β_2 is not in M^δ_α. If p decides the value of $E^{\check{G}}(\beta_1, \beta_2)$, then β_1 is either already in the range of f^p_γ, or in the set of forbidden nodes S^p_γ.

- So if $w \leqslant [p]^\delta_\alpha$, then $S^w_\gamma \supseteq S^p_\gamma \cap M$, so either w agrees with p on the node mapped to β_1, or else it cannot map anything on β_1.

We will show that every condition in $\mathbb{P}_\delta$ can be extended into a condition which is *super-nice* with respect to M^δ_α, and that whenever p is super-nice with respect to M^δ_α, then its trace is a residue of p.

Definition 3.2.4. The definition is by recursion on $\delta < \omega_2$. Let $\alpha \in \mathcal{E}_\delta$. A condition $p \in \mathbb{P}_\delta$ is **super-nice with respect to $\boldsymbol{M^\delta_\alpha}$** if

1. For every $\gamma \in \mathsf{sp}(p) \cap M^\delta_\alpha$ and $\beta \in \mathsf{ran}(f^p_\gamma) - M^\delta_\alpha$: if $\{\beta, \beta'\} \in \mathsf{dom}(E^p)$ for some $\beta' \in M^\delta_\alpha$, then $\beta' \in \mathsf{ran}(f^p_\gamma) \cup S^p_\gamma$,
2. if $\gamma \in \delta \cap M^\delta_\alpha$ and $\alpha' \in \mathsf{dom}(f^p_\gamma) - M^\delta_\alpha$, then $p \restriction \gamma$ is super-nice with respect to $M^\gamma_{\rho_\gamma(\alpha')}$.

Lemma 3.2.5. *Let $\delta < \omega_2$ and $\alpha \in \mathcal{E}_\delta$. If $p \in \mathbb{P}_\delta$ is super-nice with respect to M^δ_α and $\gamma \in \delta \cap M^\delta_\alpha$, then $p \restriction \gamma$ is super-nice with respect to M^γ_α.*

We give a technical but practical characteriztion of super-niceness in terms of finite “paths”. A **path in $\boldsymbol{p}$ from $\boldsymbol{M^\delta_\alpha}$ to $\boldsymbol{M^\gamma_\rho}$** is a finite sequence of pairs $(\gamma_0, \rho_0), \ldots, (\gamma_n, \rho_n)$ satisfying:

1. $\gamma_0 = \delta$ and $\rho_0 = \alpha$, and $\gamma_n = \gamma$ and $\rho_n = \rho$,
2. $\gamma_{k+1} \in \mathsf{sp}(p \restriction \gamma_k) \cap M^{\gamma_k}_{\rho_k}$,
3. there is $\alpha' \in \mathsf{dom}(f^p_{\gamma_{k+1}}) - \rho_k$ such that $\rho_{k+1} = \rho_{\gamma_{k+1}}(\alpha')$.

Lemma 3.2.6. *Let $\delta < \omega_2$, $p \in \mathbb{P}_\delta$ and $\alpha \in \mathcal{E}_\delta$. The following are equivalent:*

1. *p is super-nice with respect to M^δ_α,*
2. *For every path $(\gamma_0, \rho_0), \ldots, (\gamma_n, \rho_n)$ in p from M^δ_α to M^γ_ρ, it holds that for all $\xi \in \mathsf{sp}(p \restriction \gamma_n) \cap M^{\gamma_n}_{\beta_n}$: if $\beta \in \mathsf{ran}(f^p_\xi) - \rho_n$ and $\alpha < \rho_n$ are such that $\{\alpha, \beta\} \in \mathsf{dom}(E^p)$, then $\alpha \in \mathsf{ran}(f^p_\xi) \cup S^p_\xi$.*

Lemma 3.2.7. *Let $\delta < \omega_2$ and $\alpha \in \mathcal{E}_\delta$. For every $p \in \mathbb{P}_\delta$ there is $q \leqslant p$ which is super-nice with respect to M^δ_α.*

Proof. Let $p \in \mathbb{P}_\delta$. We extend p by adding all nodes in $u^p - \mathsf{ran}(f^p_\gamma)$ into S^p_γ for every $\gamma < \delta$. This is an overkill but causes no harm. Indeed, define $q \in \mathbb{P}_\delta$ by letting $q(0) := p(0)$ and for non-zero $\gamma < \delta$, let

$$q(\gamma) := (f^p_\gamma, S^p_\gamma \cup (u^p - \mathsf{ran}(f^p_\gamma)).$$

Then q is super-nice with respect to M^δ_α. In fact, then q is super-nice with respect to every structure. $\square$

We are ready to show that if p is super-nice with respect to M^δ_α, then its trace $[p]^\delta_\alpha$ is its residue into M^δ_α.

Proposition 3.2.8. *Let $\delta < \omega_2$ and $\alpha \in \mathcal{E}_\delta$. Assume that $p \in \mathbb{P}_\delta$ is super-nice with respect to M^δ_α. Then*

1. $[p]^\delta_\alpha \in \mathbb{P}_\delta \cap M^\delta_\alpha$,
2. $\forall w \in \mathbb{P}_\delta \cap M^\delta_\alpha (w \leqslant [p]^\delta_\alpha \to w \,||\, p)$.

Proof. The proof is by induction on δ.

Case $\delta = 1$:

It is clear that the trace $[p]^1_\alpha$ is a condition in $\mathbb{P}_1 \cap M^1_\alpha$. If $w \in \mathbb{P}_1 \cap M^1_\alpha$ extends the trace $[p]^\delta_\alpha$, then the pointwise union $w \cup p := (u^w \cup u^p, E^w \cup E^p, c^w \cup c^p)$ is a common extension of w and p, for all $p \in \mathbb{P}_1$ and $w \in \mathbb{P}_1 \cap M^1_\alpha$, $\alpha < \omega_1$. Note that the edge function $[u^w \cup u^p]^2 \to 2$ of a condition is only required to be a partial function. By taking a pointwise union, we do not introduce any new edges, so the function $c^w \cup c^p$ does not assign the same color to adjacent vertices.

Case δ limit:

First we show a claim about the structure of the models.

Claim 3.2.9. *If A is a finite subset of δ in M^δ_α, then there is $\gamma \in \delta \cap M^\delta_\alpha$ such that $A \subseteq \gamma \cap M^\gamma_\alpha$.*

Proof. We show that if $\xi < \gamma < \delta$ are such that $\xi, \gamma \in M^\delta_\alpha$, then $\xi \in M^\gamma_\alpha$. To this end, fix a bijection $\psi : \omega_1 \to \gamma$. (If $\gamma < \kappa$, then $\gamma < \alpha$, and there is nothing to prove.) We may assume that this bijection is $<_\theta$-minimal, so that it belongs to every elementary submodel of $(H_\theta, \in, <_\theta)$ containing γ. In particular $\psi \in M^\gamma_\alpha$. Since $\xi \in \gamma \cap M^\delta_\alpha$, there is $\beta < \alpha$ such that $\psi(\beta) = \xi$. Since $\beta, \psi \in M^\gamma_\alpha$, also $\xi = \psi(\beta) \in M^\gamma_\alpha$.

Now the claim follows by letting $\gamma := \max(A) + 1$. □

We finish the limit case.

- To see that $[p]^\delta_\alpha$ is a condition: Using Claim 3.2.9, find $\gamma \in \delta \cap M^\delta_\alpha$ such that $\mathrm{sp}([p]^\delta_\alpha) \subseteq \gamma \cap M^\gamma_\alpha$. Then also $[p \restriction \gamma]^\gamma_\alpha = [p]^\delta_\alpha \restriction \gamma$. It follows that
$$[p]^\delta_\alpha = [p \restriction \gamma]^\gamma_\alpha{}^\frown((\varnothing, \varnothing))_{\xi \in [\gamma,\delta)}.$$
Since $[p \restriction \gamma]^\gamma_\alpha$ is a condition by induction hypothesis, so must be $[p]^\delta_\alpha$.
- To see that $[p]^\delta_\alpha$ is a residue of p: Let $w \in \mathbb{P}_\delta \cap M^\delta_\alpha$ extend $[p]^\delta_\alpha$. Let $\gamma \in \delta \cap M^\delta_\alpha$ be such that $\mathrm{sp}(w) \subseteq \gamma \cap M^\delta_\alpha$. By Claim 3.2.9 we have $\mathrm{sp}(w) \subseteq M^\gamma_\alpha$. By induction hypothesis there is $q_0 \in \mathbb{P}_\gamma$ extending $w \restriction \gamma$ and $p \restriction \gamma$. Then q_0 can be extended into a condition $q \in \mathbb{P}_\delta$ extending w and p by letting
$$q(\xi) := \begin{cases} q_0(\xi) & \text{if } \xi < \gamma, \\ p(\xi) & \text{if } \xi \in [\gamma, \delta). \end{cases}$$

Case $\delta + 1$:

We argue first that $[p]^{\delta+1}_\alpha$ is a condition in $\mathbb{P}_{\delta+1} \cap M^{\delta+1}_\alpha$. Since $[p]^{\delta+1}_\alpha$ is a finite function $\delta + 1 \to M^{\delta+1}_\alpha$, it must be element of $M^{\delta+1}_\alpha$. The restriction $[p \restriction \delta]^\delta_\alpha$ is a condition in $\mathbb{P}_\delta \cap M^\delta_\alpha$ and residue of $p \restriction \delta$ into M^δ_α, by induction hypothesis and by the fact that $M^{\delta+1}_\alpha = M^\delta_\alpha$. Since
$$[p]^{\delta+1}_\alpha = [p \restriction \delta]^\delta_\alpha{}^\frown(f^{[p]^{\delta+1}_\alpha}_\delta, S^{[p]^{\delta+1}_\alpha}_\delta),$$
it suffices to show that

1. $f^p_\delta \restriction \alpha$ is a function in M^δ_α,
2. the condition $[p \restriction \delta]^\delta_\alpha$ decides edges and colors in the domain of $f^{[p]^{\delta+1}_\alpha}_\delta$ and forces that it is edge- and color-preserving.

The first item follows from Remark 3.2.2. The second item follows using the induction hypothesis - since $[p \restriction \delta]^{\delta}_{\alpha}$ is a residue of $p \restriction \delta$ into M^{δ}_{α}, it forces the same facts about $f_{\delta}^{[p]^{\delta+1}_{\alpha}}$ as $p \restriction \delta$ does. This is enough to see that $[p]^{\delta+1}_{\alpha}$ is a condition in $\mathbb{P}_{\delta+1} \cap M^{\delta+1}_{\alpha}$. There remains to show that it is a residue of p.

We then argue that $[p]^{\delta+1}_{\alpha}$ is a residue of p into $M^{\delta+1}_{\alpha}$. Let $w \in \mathbb{P}_{\delta+1} \cap M^{\delta+1}_{\alpha}$ extend $[p]^{\delta+1}_{\alpha}$. We find a common extension $q \leqslant w, p$. We already have:

- $w \restriction \delta$ and $p \restriction \delta$ are compatible, by induction hypothesis,
- $f^{w}_{\delta} \cup f^{p}_{\delta}$ is a finite injection,
- $(u^{w} \cup u^{p}, E^{w} \cup E^{p}, c^{w} \cup c^{p})$ is a condition in $\mathbb{P}_1$,
- $(S^{w}_{\delta} \cup S^{p}_{\delta}) \cap \mathsf{ran}(f^{w}_{\delta} \cup f^{p}_{\delta}) = \varnothing$.

We need to find an extension $v \leqslant w \restriction \delta, p \restriction \delta$ which decides edges in the set $\mathsf{dom}(f^{w}_{\delta}) \cup \mathsf{dom}(f^{p}_{\delta})$ such that $v(0)$ decides edges in $u^{w} \cup u^{p}$ *correctly*: in such a manner that makes $f^{w}_{\delta} \cup f^{p}_{\delta}$ edge-preserving. This condition v is found in finitely many steps, climbing up the models $\{M^{\delta}_{\rho_{\delta}(\alpha')} : \alpha' \in \mathsf{dom}(f^{p}_{\delta}) - \alpha\}$. Each model $M^{\delta}_{\rho_{\delta}(\alpha')}$ separates the node α' fromits image $f^{p}_{\delta}(\alpha')$ in the sense that

$$\alpha' \in M^{\delta}_{\rho_{\delta}(\alpha')} \quad \text{and} \quad f^{p}_{\delta}(\alpha') \notin M^{\delta}_{\rho_{\delta}(\alpha')}.$$

The point is to decide $\dot{H}_{\delta}$-edges of the set $\mathsf{dom}(f^{w}_{\delta}) \cup \{\alpha'\}$ inside the model $M^{\delta}_{\rho_{\delta}(\alpha')}$ and then extend $w(0) \cup p(0)$ accordingly outside of the model $M^{\delta}_{\rho_{\delta}(\alpha')}$, using the fact that $f^{p}_{\delta}(\alpha') \notin M^{\delta}_{\rho_{\delta}(\alpha')}$.

Let first

$$f := f^{w}_{\delta} \cup f^{p}_{\delta}.$$
$$S := S^{w}_{\delta} \cup S^{p}_{\delta}.$$

Enumerate the set $\{\alpha\} \cup \{\rho_{\delta}(\alpha') : \alpha' \in \mathsf{dom}(f^{p}_{\delta}) - \alpha\} \cup \{\omega_1\}$ in a strictly increasing order as

$$\alpha = \rho_0 < \rho_1 < \cdots < \rho_n = \omega_1.$$

Let $X_0 := \mathsf{dom}(f) \cap M^{\delta}_{\alpha} = \mathsf{dom}(f^{w}_{\delta})$ and for $k \in [1, n)$, let

$$X_k := \{\alpha' \in \mathsf{dom}(f) : f(\alpha') \in M^{\delta}_{\rho_{k+1}} - M^{\delta}_{\rho_k}\}.$$

Note that:

- $X_k \subseteq M^\delta_{\rho_k}$,
- $f[X_k] \subseteq M^\delta_{\rho_{k+1}} - M^\delta_{\rho_k}$,
- every element in $f[X_k]$ has label at most ρ_k,
- $\mathsf{dom}(f) = X_0 \cup \cdots \cup X_n$,

As said above, the goal is to find a common extension $v \leqslant w \restriction \delta, p \restriction \delta$ such that v decides edges in the set $\mathsf{dom}(f) = X_0 \cup \cdots \cup X_n$, and $v(0)$ decides edges in u exactly in such a manner that makes $f : \mathsf{dom}(f^w_\delta) \cup \mathsf{dom}(f^p_\delta) \to \dot{\mathcal{G}}$ edge-preserving. If such v can be found, we can let $q := v^\frown(f, S^w_\delta \cup S^p_\delta)$, and this will be a condition and a common extension of w and p. We find such a v in n many steps.

We now define conditions $v_k \in \mathbb{P}_\delta \cap M^\delta_{\rho_k}$ by recursion on $k < n$. Let $v_0 := w \restriction \delta$. Assume that v_k was defined and satisfies:

1. $v_k \in \mathbb{P}_\delta \cap M^\delta_{\rho_k}$,
2. v_k decides edges in the set $\mathsf{dom}(f^w_\delta) \cup X_0 \cup \cdots \cup X_{k-1}$, and for $\alpha_1, \alpha_2 \in \mathsf{dom}(f^w_\delta) \cup X_0 \cup \cdots \cup X_{k-1}$,
    - $v_k \Vdash E^{\dot{H}_\delta}(\alpha_1, \alpha_2) \quad \Longrightarrow \quad E^{v_k}(\{f(\alpha_1), f(\alpha_2)\}) = 1$,
    - $v_k \Vdash \neg E^{\dot{H}_\delta}(\alpha_1, \alpha_2) \quad \Longrightarrow \quad E^{v_k}(\{f(\alpha_1), f(\alpha_2)\}) = 0$

Up to extending v_k in the model $M^\delta_{\rho_k}$, we may assume that it also decides edges in the set $\mathsf{dom}(f^w_\delta) \cup X_k$.

We proceed in two steps: we first extend $[p \restriction \delta]^\delta_{\rho_{k+1}}$ to some $\tilde{p}$ by adding some edges to E^p, and then we obtain v_{k+1} by amalgamating v_k with $\tilde{p}$ in the model $M^\delta_{\rho_{k+1}}$.

To this end, let $\tilde{E}$ be the minimal extension of $E^p \cap M^\delta_{\rho_{k+1}}$ that satisfies that for every $\alpha_1 \in \mathsf{dom}(f^w_\delta)$ and $\alpha_2 \in X_k$:

- if $v_k \Vdash$ ``$E^{\dot{H}_\delta}(\alpha_1, \alpha_2)$, then $\tilde{E}(\{f^w_\delta(\alpha_1), f^p_\delta(\alpha_2)\}) = 1$,
- if $v_k \Vdash$ ``$\neg E^{\dot{H}_\delta}(\alpha_1, \alpha_2)$, then $\tilde{E}(\{f^w_\delta(\alpha_1), f^p_\delta(\alpha_2)\}) = 0$,

Note that $f^p_\delta(\alpha_2) \in M^\delta_{\rho_{k+1}} - M^\delta_{\rho_k}$ and furthermore, if $\alpha_1 \in \mathsf{dom}(f^w_\delta) - \mathsf{dom}(f^p_\delta)$, then the pair $\{f^w_\delta(\alpha_1), f^p_\delta(\alpha_2)\}$ is not yet in $\mathsf{dom}(E^p)$, by the assumption that p is super-nice with respect to M^δ_α. Let $\tilde{p}$ be the condition obtained from $[p \restriction \delta]^\delta_{\rho_{k+1}}$ by replacing $E^p \cap M^\delta_{\rho_{k+1}}$ by $\tilde{E}$.

Claim 3.2.10. *$\tilde{p}$ is super-nice with respect to $M^\delta_{\rho_k}$ and $[\tilde{p}]^\delta_{\rho_k} = [p \restriction \delta]^\delta_{\rho_k}$.*

Proof of Claim 3.2.10. It is clear that $[\tilde{p}]^\delta_{\rho_k} = [p \restriction \delta]^\delta_{\rho_k}$, because we only added edges where the other end lies in $M^\delta_{\rho_{k+1}} - M^\delta_{\rho_k}$. We show that $\tilde{p}$ is super-nice with respect to $M^\delta_{\rho_k}$. Suppose not. By assumption $[p \restriction \delta]^\delta_{\rho_{k+1}}$ is super-nice with respect to $M^\delta_{\rho_k}$, so by minimality of $\tilde{E}$, there is a path from $M^\delta_{\rho_k}$ to some M^γ_β and there are $\xi \in \gamma \cap M^\gamma_\beta$ and $\alpha' \in \mathsf{dom}(f^p_\gamma) - \beta$ such that $f^p_\xi(\alpha') \in f^p_\delta[X_k]$. But every node in the set $f^p_\delta[X_k]$ must have label at most ρ_k, and only nodes in ρ_k can be mapped to nodes with label ρ_k. But $\alpha' \not< \rho_k$, because $\alpha' \in \mathsf{dom}(f^p_\delta) - \beta$ and $\beta \geqslant \rho_k$. Thus α' cannot be mapped to a node in $f^p_\delta[X_k]$. Hence $\tilde{p}$ must be super-nice with respect to $M^\delta_{\rho_k}$.

□

Now $[\tilde{p}]^\delta_{\rho_k}$ is a residue of $\tilde{p}$ into $M^\delta_{\rho_k}$ and v_k extends it. Thus there is a condition

$$v_{k+1} \leqslant v_k, \tilde{p}$$

in $\mathbb{P}_\delta$. By elementarity, we may assume $v_{k+1} \in \mathbb{P}_\delta \cap M^\delta_{\rho_{k+1}}$. This ends Step $k+1$.

Finally, define

$$q := v_n {}^\frown (f, S).$$

Then q is a condition in $\mathbb{P}_{\delta+1}$ and a common extension of w and p. This ends the proof of Proposition 3.2.8.

□

Corollary 3.2.11. *For every $\delta < \omega_2$, the poset $\mathbb{P}_\delta$ is strongly proper and has ccc. The poset $\mathbb{P}_{\omega_2}$ has ccc.*

Proof. Let $\lambda > \theta$ be a regular cardinal and let $M \preccurlyeq H(\lambda)$ be countable model such that $\mathbb{P}_\delta \in M$ and $\alpha := \omega_1 \cap M \in \mathcal{E}_\delta$. It suffices to show that $1_{\mathbb{P}_\delta}$ is strongly $(\mathbb{P}_\delta, M)$-generic. We have

$$M \cap \mathbb{P}_\delta = M^\delta_\alpha \cap \mathbb{P}_\delta.$$

But since $\mathbb{P}_\delta$ is strongly proper with respect to the model M^δ_α, this implies the conclusion. □

Corollary 3.2.12. $2^\omega = \aleph_2$ *in* $V^{\mathbb{P}_{\omega_2}}$.

Proof. Each $\mathbb{P}_\delta$, $\delta < \omega_2$, is strongly proper and thus adds reals. The direction $2^\omega \leqslant \aleph_2$ follows from the fact that $\mathbb{P}_{\omega_2}$ has size $\aleph_2$. $\square$

Corollary 3.2.13. *In $V^{\mathbb{P}_{\omega_2}}$:*

1. *$\dot{\mathcal{G}}$ is a universal countably chromatic graph of size $\aleph_1$,*
2. *$2^\omega = \aleph_2$.*

We have proved:

Theorem 3.2.14. *It is consistent that the class of countaby chromatic graphs of size $\aleph_1$ has a universal object and $2^\omega = \aleph_2$.*

Chapter 4

Framework for trees and lines

This section serves to set up the path for producing a forcing extension with a universal wide μ^+-Aronszajn tree or line, when μ^+ is either $\aleph_1$ or an uncountable double successor cardinal. The two subsections are independent of each other and can be read in whichever order.

4.1 Side conditions

Let $\mathcal{K}$ be a class of objects of size μ^+ into which we plan to force a universal object. In this thesis this class will be either wide μ^+-Aronszajn trees or "locally ordered" wide μ^+-Aronszajn trees, where μ is either $\aleph_0$ or an uncountable successor cardinal. In each or the proofs in the following chapters, we will fix a weakly compact cardinal $\kappa > \mu^+$ and define an iteration $(\mathbb{P}_\delta : \delta \leqslant \kappa^+)$ that collapses κ onto μ^+ and produces the desired universal object in $\mathcal{K}$.

In this subsection we will carefully select κ^+ many stationary subsets of κ, each of which will enumerate a sequence of side conditions that are used at one stage of the iteration. The choice of the stationary sets as well as the side conditions themselves will depend on the intended poset.

Assumption 4.1.1. *Throughout the section, we fix a weakly compact cardinal κ.*

Notation 4.1.2. Fix a regular cardinal $\theta > (2^\kappa)^+$ and a well-order $<_\theta$ of H_θ.

Recall the normal weakly compact filter $\mathcal{F}_{\mathrm{wc}}$ from Lemma 2.2.2.

Definition 4.1.3. Let $\alpha < \kappa$. A model $M \preccurlyeq H_\theta$ **reflects at $\boldsymbol{\alpha}$** if for every $A \in \mathscr{P}(V_\kappa) \cap M$ and Π^1_1-formula φ,

$$V_\kappa \models \varphi(A) \implies V_\alpha \models \varphi(A \cap V_\alpha).$$

Since κ is weakly compact, there are many models that reflect at α for many $\alpha < \kappa$ (See Lemma 4.1.5 below).

We follow the convention that $\mathbb{P}_0 = \{\varnothing\}$. For a set of ordinals X, we denote by $\lim X$ the set of limit points of X. We now assume that we are in the middle of an inductive definition and have defined posets $\mathbb{P}_\gamma \in H_\theta$, $\gamma \leqslant \delta$.

Definition 4.1.4. By recursion on $\delta < \kappa^+$: whenever the poset $\mathbb{P}_\delta$ and the sets $\bar{\mathcal{E}}_\delta := (\mathcal{E}_\gamma)_{\gamma<\delta}$ have been defined:

1. For every $\alpha \leqslant \kappa$, let
$$M^\delta_\alpha := \text{Skolem hull of } \alpha \cup \{\kappa, \delta, \mathbb{P}_\delta, \bar{\mathcal{E}}_\delta\} \text{ in } (H_\theta, \in, <_\theta),$$
2. Let
$$\begin{aligned} \mathcal{E}_\delta := \{\alpha < \kappa : \ & (1) \quad V_\kappa \cap M^\delta_\alpha = V_\alpha, \\ & (2) \quad M^\delta_\alpha \text{ reflects at } \alpha, \\ & (3) \quad \alpha \in \bigcap_{\gamma \in \delta \cap M^\delta_\alpha} \lim \mathcal{E}_\gamma\}. \end{aligned}$$

Note that the case $\alpha = \kappa$ is allowed in the definition on M^δ_α.

Lemma 4.1.5.

1. *The set $\mathcal{E}_\delta$ belongs to the weakly compact filter $\mathcal{F}_{\mathrm{wc}}$ for every $\delta < \kappa^+$.*
2. *For every $\delta < \kappa^+$ and $\alpha \in \mathcal{E}_\delta$:*
    - *(a) $\mathbb{P}_\delta \in M^\delta_\alpha$,*
    - *(b) $\mathbb{P}_\gamma \in M^\delta_\alpha$ for every $\gamma \in \delta \cap M^\delta_\alpha$,*
    - *(c) $M^{\delta'}_{\alpha'} \in M^\delta_\alpha$ for every $\alpha' < \alpha$ and $\delta' \in (\delta + 1) \cap M^\delta_\alpha$,*
    - *(d) $\delta \cap M^\delta_\alpha = \delta \cap M^{\delta+1}_\alpha$,*
    - *(e) if $\xi < \gamma < \delta$ and $\xi, \gamma \in M^\delta_\alpha$, then $\xi \in M^\gamma_\alpha$.*

Proof.

1. This follows from the assumption that κ is weakly compact and the normality of the filter $\mathcal{F}_{\text{wc}}$.

2. For item (2c): if $\beta < \alpha$ and $\gamma \in M^\delta_\alpha$, then $M^\gamma_\beta \subseteq M^\delta_\alpha$, and since M^δ_α is closed under sequences of length $< \alpha$ and $|M^\gamma_\beta| = \beta < \alpha$, we obtain $M^\gamma_\beta \in M^\delta_\alpha$. For item (2d): if $\delta < \kappa$, then $\delta < \alpha$ and the claim follows easily, and otherwise if $\psi : \kappa \to \delta$ is the $<_\theta$-least bijection, then $\delta \cap M^{\delta+1}_\alpha = \psi[\alpha] = \delta \cap M^\delta_\alpha$. Item (2e) is proved similarly: if $\gamma < \kappa$, then $\gamma < \alpha$ and the claim follows easily, and otherwise if $\psi : \kappa \to \gamma$ is the $<_\theta$-least bijection, then $\psi \in M^\delta_\alpha$ and $\psi \in M^\gamma_\alpha$, which implies $\gamma \cap M^\delta_\alpha = \psi[\alpha] = \gamma \cap M^\gamma_\alpha$, and gives directly $\xi \in M^\gamma_\alpha$.

□

Lemma 4.1.6. *If $\mathbb{P} \subseteq V_\kappa$ and $\mathbb{P}$ is strongly proper with respect to the models $\{M^\delta_\alpha : \alpha \in \mathcal{E}_\delta\}$ for some $\delta < \kappa^+$, then $\mathbb{P}$ is κ-strongly proper.*

Proof. Let $\lambda > \theta$ be a regular cardinal. As $\mathcal{E}_\delta \in \mathcal{F}_{\text{wc}}$, there are stationarily many models M in $\mathscr{P}_\kappa(H_\lambda)$ that are elementary in H_λ and satisfy $M \cap V_\kappa = V_\alpha$ for some $\alpha \in \mathcal{E}_\delta$. Fix one such model M. Now $M \cap V_\kappa = V_\alpha$ and the assumption $\mathbb{P} \subseteq V_\kappa$ imply

$$\mathbb{P} \cap M = \mathbb{P} \cap M^\delta_\alpha.$$

It follows that every strongly $(\mathbb{P}, M^\delta_\alpha)$-generic condition is $(\mathbb{P}, M)$-generic, and thus strong properness of $\mathbb{P}$ with respect to M^δ_α implies strong properness of $\mathbb{P}$ with respect to M. □

For a model $M \preccurlyeq H_\theta$ and a generic filter G on a poset $\mathbb{P} \in M$, we denote

$$\boldsymbol{M[G]} = \{\dot{a}^G : \dot{a} \in V^{\mathbb{P}} \cap M\}.$$

Lemma 4.1.7. *If G is a generic filter on $\mathbb{P}_\delta$ and $\alpha \in \mathcal{E}_\delta$, then $M^\delta_\alpha[G]$ is closed under $< \alpha$-sequences and elementary in $H_\theta[G]$.*

Proof. The fact that $M^\delta_\alpha[G]$ is elementary in $H_\theta[G]$ follows using the Tarski-Vaught criterion and the Maximality principle: if $H_\theta[G] \models \exists x \varphi(x, \dot{a}^G)$ for some $\dot{a}^G \in M[G]$, then there is $\dot{b}$ such that $\Vdash \exists x \varphi(x, \dot{a}) \to \varphi(\dot{b}, \dot{a})$, and by elementarity $M^\delta_\alpha \preccurlyeq H_\theta$, such $\dot{b}$ can be found in M^δ_α. Then $H_\theta[G] \models \varphi(\dot{b}^G, \dot{s}^G)$ where $\dot{b}^G \in M^\delta_\alpha[G]$ and we are done.

To see that $M^\delta_\alpha[G]$ is closed under $< \alpha$-sequences, let $\bar\alpha < \alpha$ and let $a : \bar\alpha \to M^\delta_\alpha[G]$ be a function. We show that $a \in M^\delta_\alpha[G]$. We may assume $\mathrm{ran}(a) \subseteq \mathrm{Ord}$. For each $i < \bar\alpha$, let $\dot a_i \in V^{\mathbb{P}_\delta} \cap M^\delta_\alpha$ be a name for $a(i)$. For every $i < \bar\alpha$ there is some $\alpha_i < \alpha$ and a condition $p_i \in G \cap V_{\alpha_i}$ that decides $\dot a_i$, i.e. forces $\dot a_i = \check a(i)$. It follows that there is a name $\dot b \in V^{\mathbb{P}_\delta}$ such that $\dot b^G = \dot a^G$ and $\dot b \subseteq V_{\sup_{i<\bar\alpha} \alpha_i}$. The ordinal α is an inaccessible cardinal because $\alpha \in \mathcal{E}_\delta$, so in particular $\sup_{i<\bar\alpha} \alpha_i < \alpha$, and thus $\dot b \in M^\delta_\alpha$. It follows that $\dot b^G = a \in M^\delta_\alpha[G]$. Hence $M^\delta_\alpha[G]$ is closed under $< \alpha$-sequences. $\square$

We end the subsection with a technical discussion of the structure of the collection of the side conditions M^δ_α, $\alpha \in \mathcal{E}_\delta$, $\delta < \kappa^+$.

Definition 4.1.8. Let $\delta \leqslant \kappa^+$ and $\alpha, \beta \leqslant \kappa$. A **path** from M^δ_α to M^γ_β is a finite sequence of pairs

$$(\gamma_0, \beta_0), \ldots, (\gamma_n, \beta_n)$$

such that $(\gamma_0, \beta_0) = (\delta, \alpha)$ and $(\gamma_n, \beta_n) = (\gamma, \beta)$, and the following are satisfied:

1. $\gamma_{k+1} < \gamma_k$ and $\gamma_{k+1} \in M^{\gamma_k}_{\beta_k}$,
2. $\beta_{k+1} > \beta_k$ and $\beta_{k+1} \in \mathcal{E}_{\gamma_{k+1}}$.

Lemma 4.1.9.

1. *If there is a path from M^δ_α to M^γ_β, then $\gamma \cap M^\delta_\alpha \subseteq M^\gamma_\beta$.*
2. *If there is a path from M^δ_α to M^γ_β and and $\beta \leqslant \alpha^+$, then $\gamma \in M^\delta_{\alpha^+}$, where α^+ is the next element of $\mathcal{E}_\delta$ strictly above α.*

Proof. We prove item (1). The proof is by induction on the length of the path. To this end, fix a path

$$(\gamma_0, \beta_0), \ldots, (\gamma_{n+1}, \beta_{n+1}).$$

with $(\gamma_0, \beta_0) = (\delta, \alpha)$. We claim that

$$\gamma_{n+1} \cap M^\delta_\alpha \subseteq M^{\gamma_{n+1}}_{\beta_{n+1}}.$$

Let $\xi \in \gamma_{n+1} \cap M^\delta_\alpha$. Let ψ be the $<_\theta$-least bijection from κ to γ_{n+1}. Then $\psi \in M^{\gamma_{n+1}}_{\beta_{n+1}}$ but also $\psi \in M^{\gamma_n}_{\beta_n}$ since $\gamma_{n+1} \in M^{\gamma_n}_{\beta_n}$. By the induction hypothesis we have $\gamma_n \cap M^\delta_\alpha \subseteq M^{\gamma_n}_{\beta_n}$, so $\xi \in M^{\gamma_n}_{\beta_n}$ too. And thus $\bar\xi := \psi^{-1}(\xi) \in \beta_n \subseteq \beta_{n+1}$. And then $\xi = \psi(\bar\xi) \in M^{\gamma_{n+1}}_{\beta_{n+1}}$. The fact that $\mathbb{P}_\gamma \cap M^\delta_\alpha \subseteq \mathbb{P}_\gamma \cap M^\gamma_\beta$ follows immediately.

The proof of item (2) is proved the same way by induction on the length of the path, using appropriate injection ψ. $\square$

Definition 4.1.10. Let $\delta < \kappa^+$ and $\alpha \in \mathcal{E}_\delta$. The **closure** of M^δ_α is the set of pairs (γ, β) such that there is a path from M^δ_α to M^γ_β.

We tacitly identify each closure with its lexicographic ordering: $(\gamma, \beta) < (\gamma', \beta')$ if $\beta < \beta'$ or if $\beta = \beta'$ and $\gamma < \gamma'$. It holds that if $((\gamma_j, \beta_j) : j < \iota)$ is the lexicographic order of a closure, then it satisfies that for any $j > 0$ there is $i < j$ with $\gamma_j < \gamma_i$ and $\gamma_j \in M^{\gamma_i}_{\beta_i}$. We abstract this property:

Definition 4.1.11. Let $\delta < \kappa^+$ and $\alpha \in \mathcal{E}_\delta$. A **$\boldsymbol{\delta}$-sequence** is a sequence $((\delta_j, \alpha_j) : j < \iota)$ indexed by an ordinal ι that satisfies the following:

1. $\alpha_j \in \mathcal{E}_{\delta_j}$ for every $j < \iota$,
2. $\delta_j \leqslant \delta$ for every $j < \iota$,
3. $\alpha_i \leqslant \alpha_j$ for every $i < j < \iota$,
4. for every $j > 0$, there is $i < j$ such that $\delta_j \leqslant \delta_i$ and $\delta_j \in M^{\delta_i}_{\alpha_i}$.

Lemma 4.1.12. *The closure of a model M^δ_α (identified with its lexicographic ordering) is always a δ-sequence.*

Proof. Clear from definitions. □

And every δ-sequence is contained in the closure of its root model:

Lemma 4.1.13. *If $((\delta_j, \alpha_j) : j < \iota)$ is a δ-sequence, then there is a path from $M^{\delta_0}_{\alpha_0}$ to $M^{\delta_j}_{\alpha_j}$ for every $j < \iota$.*

Proof. Straightforward induction on the length of the δ-sequence, using item (4) of the definition of δ-sequence. □

Lemma 4.1.14. *Let $((\delta_i, \alpha_i) : i \leqslant n)$ and $((\gamma_j, \beta_j) : j \leqslant m)$ be paths with the same root $M^{\delta_0}_{\alpha_0} = M^{\gamma_0}_{\beta_0}$. If $\alpha_n \leqslant \beta_m$, then*

$$(\min\{\delta_n, \gamma_m\} + 1) \cap M^{\delta_n}_{\alpha_n} \subseteq M^{\gamma_m}_{\beta_m}.$$

Proof. We prove the lemma by induction on $n + m$. The proof is split into cases. The case when either $\delta_n < \kappa$ or $\gamma_m < \kappa$ is vacuously true, so we assume $\delta_n, \gamma_m \geqslant \kappa$. For an ordinal $\delta \geqslant \kappa$, we denote by ψ_δ the $<_\theta$-least bijection $\delta \to \kappa$. Clearly it holds that if $\delta \in M^\gamma_\beta$, then $\psi_\delta \in M^\gamma_\beta$.

Case 1: $\alpha_n \leqslant \beta_{m-1}$.

We have

$$\begin{aligned}
&(\min\{\gamma_m, \delta_n\} + 1) \cap M^{\delta_n}_{\alpha_n} \\
&\subseteq (\min\{\gamma_m, \delta_n\} + 1) \cap (\min\{\gamma_{m-1}, \delta_n\} + 1) \cap M^{\delta_n}_{\alpha_n} && \text{(a)} \\
&\subseteq (\min\{\gamma_m, \delta_n\} + 1) \cap M^{\gamma_{m-1}}_{\beta_{m-1}} && \text{(b)} \\
&\subseteq (\gamma_m + 1) \cap M^{\gamma_{m-1}}_{\beta_{m-1}} && \text{(c)} \\
&\subseteq M^{\gamma_m}_{\beta_m} && \text{(d)}
\end{aligned}$$

Inclusion (a) follows from the fact that $\gamma_m < \gamma_{m-1}$. Inclusion (b) follows from induction hypothesis using the case assumption $\alpha_n \leqslant \beta_{m-1}$. Inclusion (c) follows from the fact that $\min\{\gamma_m, \delta_n\} + 1 \leqslant (\gamma_m + 1)$. Inclusion (d) follows from the fact that $\gamma_m \in M^{\gamma_{m-1}}_{\beta_{m-1}}$ and from Lemma 4.1.9(1).

Case 2: $\beta_{m-1} < \alpha_n \leqslant \beta_m$ and $\gamma_m \leqslant \delta_n$.

By the case assumption $\gamma_m \leqslant \delta_n$, the claim is now $(\gamma_m + 1) \cap M^{\delta_n}_{\alpha_n} \subseteq M^{\gamma_m}_{\beta_m}$. We have $\gamma_m \in M^{\gamma_m}_{\beta_m}$ by definition of $M^{\gamma_m}_{\beta_m}$. We will show that $\gamma_m \cap M^{\delta_n}_{\alpha_n} \subseteq M^{\gamma_m}_{\beta_m}$. Since $\beta_{m-1} \leqslant \alpha_n$, the induction hypothesis becomes

$$(\min\{\gamma_{m-1}, \delta_n\} + 1) \cap M^{\gamma_{m-1}}_{\beta_{m-1}} \subseteq M^{\delta_n}_{\alpha_n}.$$

The case assumption implies $\gamma_m \leqslant \min\{\gamma_{m-1}, \delta_n\}$, so it follows from the induction hypothesis that $\gamma_m \in M^{\delta_n}_{\alpha_n}$. There remains to see that $\gamma_m \cap M^{\delta_n}_{\alpha_n} \subseteq M^{\gamma_m}_{\beta_m}$: Let $\xi \in \gamma_m \cap M^{\delta_n}_{\alpha_n}$. Let ν be such that $\psi_{\gamma_m}(\xi) = \nu$. Since $\gamma_m \in M^{\delta_n}_{\alpha_n}$, it follows that $\psi_{\gamma_m}, \xi \in M^{\delta_n}_{\alpha_n}$, and thus $\nu = \psi_{\gamma_m}(\xi) < \alpha_n$. Now $\alpha_n \leqslant \beta_m$, so $\nu \in M^{\gamma_m}_{\beta_m}$. Also $\gamma_m \in M^{\gamma_m}_{\beta_m}$, so $\xi = \psi^{-1}_{\gamma_m}(\nu) \in M^{\gamma_m}_{\beta_m}$.

Case 3: $\beta_{m-1} < \alpha_n \leqslant \beta_m$ and $\delta_n < \gamma_m$.

By the case assumption the claim is

$$(\delta_n + 1) \cap M^{\delta_n}_{\alpha_n} \subseteq M^{\gamma_m}_{\beta_m}.$$

We have $\alpha_{n-1} \leqslant \alpha_n \leqslant \beta_m$, so the induction hypothesis implies

$$(\min\{\delta_{n-1}, \gamma_m\} + 1) \cap M^{\delta_{n-1}}_{\alpha_{n-1}} \subseteq M^{\gamma_m}_{\beta_m}.$$

The case assumption $\delta_n < \gamma_m$ implies $\delta_n < \min\{\delta_{n-1}, \gamma_m\}$, so the induction hypothesis gives $\delta_n \in M^{\gamma_m}_{\beta_m}$.

There remains to show that $\delta_n \cap M^{\delta_n}_{\alpha_n} \subseteq M^{\gamma_m}_{\beta_m}$. Let $\xi \in \delta_n \cap M^{\delta_n}_{\alpha_n}$. Let ν be such that $\psi_{\delta_n}(\xi) = \nu$. Then $\xi, \delta_n \in M^{\delta_n}_{\alpha_n}$ imply $\nu < \alpha_n \leqslant \beta_m$. Thus $\nu \in M^{\gamma_m}_{\beta_m}$. By the above paragraph we have $\delta_n \in M^{\gamma_m}_{\beta_m}$, so

$$\xi = \psi^{-1}_{\delta_n}(\nu) \in M^{\gamma_m}_{\beta_m}.$$

We have proved the lemma.

$\square$

Lemma 4.1.15. *Let $\delta < \kappa^+$. If $((\delta_j, \alpha_j) : j < \iota)$ is a δ-sequence, then for all $i < j < \iota$,*

$$(\min\{\delta_i, \delta_j\} + 1) \cap M^{\delta_i}_{\alpha_i} \subseteq M^{\delta_j}_{\alpha_j}.$$

In particular, if $\delta_i \leqslant \delta_j$, then $\delta_i \in M^{\delta_j}_{\alpha_j}$.

Proof. Every δ-sequence consists of paths from its root condition $M^{\delta_0}_{\alpha_0}$, by Lemma 4.1.13, so the lemma follows from Lemma 4.1.14. $\square$

In order to be able to apply induction hypotheses in the proofs of later lemmas, we need to be able to transform a δ-sequence into a δ'-sequence for those $\delta' < \delta$ that occur in the sequence, in some natural way. Furthermore, we need to be able to do this in the following scenario: Suppose that $((\delta_j, \alpha_j) : j < \iota)$ is a δ-sequence and $j < \iota$ is a fixed index. We want to simlutaneously project the sequence to be an δ_j-sequence, take (a cofinal subsequence of) its initial segment, and restrict it so that it will be an element in the model $M^{\delta_j}_{\alpha_j}$. The following lemma allows to do this.

Lemma 4.1.16. *Let $\delta < \kappa^+$. If $((\delta_j, \alpha_j) : j < \iota)$ is a δ-sequence and $j < \iota$, then there is $k < j$ such that the restricted sequence*

$$((\min\{\delta_i, \delta_j\}, \alpha_i) : k \leqslant i < j)$$

is a δ_j-sequence. If furthermore $M^{\delta_j}_{\alpha_i}$ is closed under $\leqslant j$-sequences, then the restricted sequence is an element in $M^{\delta_j}_{\alpha_j}$.

Proof. Denote

$$\delta^j_i := \min\{\delta_i, \delta_j\}.$$

Let $k < j$ be such that $\delta_j \leqslant \delta_k$ and $\delta_j \in M^{\delta_k}_{\alpha_k}$. We show first that the sequence

$$((\delta^j_i, \alpha_i) : k \leqslant i < j)$$

is an element in $M^{\delta_j}_{\alpha_j}$, in case $M^{\delta_j}_{\alpha_j}$ is closed under $\leqslant j$-sequences. It suffices to show that $\delta^j_i \in M^{\delta_j}_{\alpha_j}$ for every $i \in [k, j)$, and the conclusion follows as $M^{\delta_j}_{\alpha_j}$ is closed under $\leqslant j$-sequences. But this follows by Lemma 4.1.15:

$$\delta^j_i \in (\min\{\delta_i, \delta_j\} + 1) \cap M^{\delta_i}_{\alpha_i} \subseteq M^{\delta_j}_{\alpha_j}.$$

We are ready to show that the restricted sequence is a δ_j-sequence. It suffices to verify items (1) and (4) of the definition of δ-sequence.

We verify item (1) of Definition 4.1.11. Let $i \in [k, j)$. If $\delta_i < \delta_j$, then $\delta^j_i = \delta_i$ and so $\alpha_i \in \mathcal{E}_{\delta_i}$ by assumption. Thus suppose that $\delta_j < \delta_i$. To show $\alpha_i \in \mathcal{E}_{\delta_j}$, it suffices to show $\delta_j \in M^{\delta_i}_{\alpha_i}$. This follows by an application of Lemma 4.1.15:

$$\delta_j \in (\min\{\delta_k, \delta_i\} + 1) \cap M^{\delta_k}_{\alpha_k} \subseteq M^{\delta_i}_{\alpha_i}.$$

Thus $\alpha_i \in \mathcal{E}_{\delta^j_i}$ for every $i \in [k, j)$.

We verify item (4) of Definition 4.1.11, i.e. we show that for any $i \in (k, j)$ there is $l \in [k, i)$ such that $\delta^j_i \leqslant \delta^j_l$ and $\delta^j_i \in M^{\delta^j_l}_{\alpha_l}$. Fix $i \in (k, j)$. If $\delta_i \geqslant \delta_j$, then $\delta^j_i = \delta_j$, and so $l := k$ is as wanted because $\delta_j \leqslant \delta_k$ and $\delta_j \in M^{\delta_k}_{\alpha_k}$. Suppose thus that $\delta_i < \delta_j$. Then $\delta^j_i = \delta_i$. Since we started with a δ-sequence, there is $i' < i$ such that $\delta_i \leqslant \delta_{i'}$ and $\delta_i \in M^{\delta_{i'}}_{\alpha_{i'}}$. We split the proof into two cases.

- If $i' \leqslant k$, then we argue that $l := k$ is as wanted, i.e. satisfies $\delta_i \leqslant \delta^j_k$ and $\delta_i \in M^{\delta^j_k}_{\alpha_k}$. Note that $\delta^j_k = \delta_j$. Thus the part $\delta_i \leqslant \delta^j_k$ is fine since we assumed $\delta_i < \delta_j$. There remains to see that $\delta_i \in M^{\delta_j}_{\alpha_k}$. Lemma 4.1.15 gives

  $$\delta_i \in (\min\{\delta_{i'}, \delta_k\} + 1) \cap M^{\delta_{i'}}_{\alpha_{i'}} \subseteq M^{\delta_k}_{\alpha_k}.$$

  Thus we have $\delta_i, \delta_j \in M^{\delta_k}_{\alpha_k}$. Since $\delta_i < \delta_j$, we also have $\delta_i \in M^{\delta_j}_{\alpha_k}$: if $\psi_{\delta_j} : \delta_j \to \kappa$ is the $<_\theta$-least bijection and ν is such that $\psi_{\delta_j}(\delta_i) = \nu$, then $\nu < \alpha_k$, because $\psi_{\delta_j}, \delta_i, \nu \in M^{\delta_k}_{\alpha_k}$, and thus $\delta_i = \psi^{-1}_{\delta_j}(\nu) \in M^{\delta_j}_{\alpha_k}$.

- Suppose that $k < i'$. We argue that $l := i'$ is as wanted, i.e. $\delta_i \leqslant \delta^j_{i'}$ and $\delta_i \in M^{\delta^j_{i'}}_{\alpha_{i'}}$. Again the part $\delta_i \leqslant \delta^j_{i'}$ is fine.
  - If $\delta_{i'} \leqslant \delta_j$, then $\delta^j_{i'} = \delta_{i'}$, and so clearly $\delta_i \in M^{\delta_{i'}}_{\alpha_{i'}} = M^{\delta^j_{i'}}_{\alpha_{i'}}$.
  - If $\delta_j < \delta_{i'}$, we need to show that $\delta_i \in M^{\delta_j}_{\alpha_{i'}}$. We again apply Lemma 4.1.15 and obtain

    $$\delta_j \in (\min\{\delta_k, \delta_{i'}\} + 1) \cap M^{\delta_k}_{\alpha_k} \subseteq M^{\delta_{i'}}_{\alpha_{i'}}.$$

> Thus we have $\delta_i, \delta_j \in M^{\delta_{i'}}_{\alpha_{i'}}$. Since $\delta_i < \delta_j$, we have $\delta_i \in M^{\delta_j}_{\alpha_{i'}}$: if ν is such that $\psi_{\delta_j}(\delta_i) = \nu$, then $\nu < \alpha_{i'}$, and thus $\delta_i = \psi^{-1}_{\delta_j}(\nu) \in M^{\delta_j}_{\alpha_{i'}}$.

This ends the proof of the lemma. □

4.2 Forcing a flexible tree

Let $\mu < \kappa$ be regular cardinals. In this subsection we define a $\mathsf{Col}(\mu, < \kappa)$-name $\dot{T}$ for a wide κ-Aronszajn tree. In later constructions, $\dot{T}$ will serve as a "flexible tree" into which it is easy, with further forcing, to embed other wide κ-trees.

Proposition 4.2.1. *Let $\mu < \kappa$ be regular cardinals. There is a $\mathsf{Col}(\mu, < \kappa)$-name $\dot{T}$ for a tree such that the following are satisfied:*

1. *In $V^{\mathsf{Col}(\mu,<\kappa)}$, $\dot{T}$ is a normal wide $<\mu$-closed κ-Aronszajn tree such that any node $t \in T$ has κ many successors at any level above.*
2. *Each α-th level of $\dot{T}$ is the set $\mathsf{Lev}_\alpha(\dot{T}) = \kappa \times \{\alpha\}$.*
3. *There are club many $\alpha < \kappa$ such that for any generic filter $G \subseteq \mathsf{Col}(\mu, < \kappa)$,*
   $$\dot{T}^G \cap V_\alpha \in V[G \cap V_\alpha].$$
4. *There is a function $\mathsf{wd} : T \to \kappa$ in V such that for any node $t \in T$ at a level of cofinality at least μ, $\mathsf{wd}(t)$ is forced to be the least ordinal $\mathsf{wd}(t) \geqslant \mathsf{ht}(t)$ such that the branch below t is contained in the set $\mathsf{wd}(t) \times \mathsf{ht}(t)$. For any $\alpha \leqslant \beta < \kappa$, where β is inaccessible, there are κ many nodes $t \in \mathsf{Lev}_\alpha(T)$ with $\mathsf{wd}(t) = \beta$. The ordinal $\mathsf{wd}(t)$ is called the **width of t**.*
5. (Node projections.) *For every node $t \in T$ and a condition $p \in \mathsf{Col}(\mu, < \kappa)$, the set*
   $$\{t' \in T : p \Vdash ``t' <_{\dot{T}} t"\}$$
   *has size $< \mu$. We call the $<_{\dot{T}}$-supremum of this set the **node projection of t with respect to p** and let $\dot{\pi}^p(t)$ be a $\mathsf{Col}(\mu, < \kappa)$-name for this supremum.*
6. (Flexibility.) *For $t \in T$ at a level of cofinality at least μ, we say that a condition $p \in \mathsf{Col}(\mu, < \kappa)$ is **nice with respect to t**, if for any $\alpha < \kappa$ with $t \in T - (\alpha \times \alpha)$ and any $\bar{t} \in \mathsf{wd}(t) \times \mathsf{ht}(t)$, there is a condition $q \leqslant p$ such that*

  (a) $q \cap V_\alpha = p \cap V_\alpha$,
  (b) q *is a minimal extension of* p *in the sense that if* $t' \neq t$*, then*

$$\dot{\pi}^q(t') = \dot{\pi}^p(t'),$$

  (c) *any extension of* q *that forces* ``$\dot{\pi}^p(t) <_{\dot{T}} \bar{t}$" *also forces* ``$\bar{t} <_{\dot{T}} t$".

It holds for any set of nodes $A \subseteq T$ *that the set of conditions that are nice with respect to every* $t \in A$ *is dense in* $\mathsf{Col}(\mu, < \kappa)$.

7. (Labels.) *There is a function* $\ell : T \to \kappa$ *with the following properties:*
   (a) $\mathsf{wd}(t) \leqslant \ell(t)$ *for every* $t \in T$,
   (b) *for any* $\alpha \leqslant \beta_0 \leqslant \beta_1 < \kappa$*, where* $\mathsf{cf}(\alpha), \mathsf{cf}(\beta_0) \geqslant \mu$ *there are* κ *many nodes* $t \in \mathsf{Lev}_\alpha(T)$ *with* $\mathsf{wd}(t) = \beta_0$ *and* $\ell(t) = \beta_1$.

The ordinal $\boldsymbol{\ell(t)}$ *is called the* ***label of*** $\boldsymbol{t}$.

Proof. We build a $\mathsf{Col}(\mu, < \kappa)$-name $\dot{T}$ for a wide κ-Aronszajn tree on a set of size κ. We let the domain the tree $\dot{T}$ to be the $T := \kappa \times \kappa$. The αth level will be the set $\kappa \times \{\alpha\}$, for each $\alpha < \kappa$. For $t \in T$, we denote by α_t and β_t the ordinals such that $t = (\beta_t, \alpha_t)$. The ordinal α_t will be the height of the node t. The tree-order $<_{\dot{T}}$ on T will be defined in $V^{\mathsf{Col}(\mu, <\kappa)}$.

First, we fix two functions $\mathsf{wd} : T \to \kappa$ and $\ell : T \to \kappa$ (in V) with the following properties:

1. $\mathsf{wd}(t)$ and $\ell(t)$ are inaccessible cardinals and for any $t \in T$,

$$\alpha_t \leqslant \mathsf{wd}(t) \leqslant \ell(t) \leqslant \text{the least inaccessible above } \max\{\alpha_t, \beta_t\},$$

2. if $\alpha \leqslant \beta_0 \leqslant \beta_1 < \kappa$ where β_0 and β_1 are inaccessible cardinals, then there are κ many nodes $t \in \mathsf{Lev}_\alpha(T)$ with $\mathsf{wd}(t) = \beta_0$ and $\ell(t) = \beta_1$.

Then, denote by $\mathbb{Q}$ the poset consisting of finite partial functions $p : T \times \mu \to T$ such that if $(t, \gamma) \in \mathsf{dom}(p)$, then

$$f_t^p(\gamma) := p(t, \gamma) \in \mathsf{wd}(t) \times \mathsf{ht}(t).$$

The order of $\mathbb{Q}$ is inverse inclusion. A generic filter $G \subseteq \mathbb{Q}$ yields, for every $t \in T$, a surjection

$$f_t^G : \mu \to \mathsf{wd}(t) \times \mathsf{ht}(t).$$

It can be seen that $\mathbb{Q}$ is isomorphic to $\mathsf{Col}(\mu, <\kappa)$.

By the isomorphism, it suffices to define the tree order $<_{\dot{T}}$ in $V^{\mathbb{Q}}$ instead of $V^{\mathsf{Col}(\mu,<\kappa)}$. To this end, let $G \subseteq \mathbb{Q}$ be a generic filter. Working in $V[G]$, we define the order $<_T$ on T by recursion on levels $\alpha < \kappa$. Suppose that the order has already been defined on $\mathsf{Lev}_{<\alpha}(T)$. Suppose inductively that $(\mathsf{Lev}_{<\alpha}(T), <_T)$ is $<\mu$-closed and that every node has κ many successors at any level below α.

If α is a successor ordinal, say $\alpha = \alpha' + 1$, then, using a pairing function (in V), we assign κ many immediate successors in $\mathsf{Lev}_\alpha(T)$ to every node in $\mathsf{Lev}_{\alpha'}(T)$. This can be done in such a way that if t is the immediate successor of t', then $t' \in \mathsf{wd}(t) \times \mathsf{ht}(t)$, for any $t, t' \in \mathsf{Lev}_{\leqslant\alpha}$.

At a limit stage α, we assign a branch below t, denoted by b_t, to each node $t \in \mathsf{Lev}_\alpha(T)$. This suffices, since then we let $s <_T t$ if $s \in b_t$. We consider the limit cases of cofinality $< \mu$ and of cofinality $\geqslant \mu$ separately.

Suppose that α is a limit ordinal of cofinality $< \mu$. Then, in order for T to be $<\mu$-closed, we need to let every branch go through. We do this by enumerating the branches, as follows. For every branch b that is a cofinal branch through $\mathsf{Lev}_{<\alpha}(T)$, let $\beta(b)$ be the least ordinal that is an inaccessible cardinal in V such that $b \subseteq \beta(b) \times \alpha$. Let A be the range of the function $b \mapsto \beta(b)$. Then $|A| = \kappa$ and for every $\beta' \in A$, there are κ many branches b with $\beta(b) = \beta'$. Enumerate them as $\{b_{\gamma,\beta} : \gamma < \kappa\}$. Up to re-arranging, we may assume that $\mathsf{Lev}_\alpha(T) = \{t : \alpha_t = \alpha \text{ and } \mathsf{wd}(t) \in A\}$. Now for each $\beta \in A$, enumerate the nodes in $\mathsf{Lev}_\alpha(T)$ of width β as $\{t_{\beta,\gamma} : \gamma < \kappa\}$. Then we have a bijection from nodes in $\mathsf{Lev}_\alpha(T)$ to branches through $\mathsf{Lev}_{<\alpha}(T)$ defined by $b_{t_{\beta,\gamma}} := b_{\beta,\gamma}$.

Suppose then that α is an ordinal of cofinality at least μ. Given a node $t \in \mathsf{Lev}_\alpha(T)$, the branch b_t is derived from the generic collapse function

$$f_t^G := \bigcup_{p \in G} p(t, \cdot) : \mu \to \mathsf{wd}(t) \times \mathsf{ht}(t),$$

by recursion on $i < \mu$ as follows.

- Let t_0 be $f_t^G(0)$.
- If t_i was defined, we look for the least $j \in [i, \mu)$ such that the node $f_t^G(j)$ is strictly above t_i in the tree order $<_T \restriction \mathsf{Lev}_{<\alpha}(T)$ and we let $t_{i+1} := f_t^G(j)$.
- If $j < \mu$ is a limit ordinal, let t_j be the $<_T$-supremum of the chain $(t_i : i < j)$.

The branch below t can now be defined to be the set

$$b_t := \bigcup_{i<\mu} \{s \in \mathsf{Lev}_{<\alpha}(T) : s <_T t_i\}.$$

It follows by a density argument that b_t is a cofinal branch through $\mathsf{Lev}_{<\alpha}(T)$. If $t \neq t'$, then the functions f_t^G and $f_{t'}^G$ are mutually generic, which implies that the branches b_t and $b_{t'}$ will be different. This concludes the definition of the tree $(T, <_T)$. Let $<_{\dot{T}}$ be a $\mathbb{Q}$-name for the order $<_T$ and denote $\dot{T} = (T, <_{\dot{T}})$. It remains to verify that $\dot{T}$ is as wanted.

Claim 4.2.2. *The tree $\dot{T}$ satisfies items (1)-(7) of the proposition.*

Proof of Claim 4.2.2.

Items (1)-(4). It is clear that in $V^{\mathbb{Q}}$, $\dot{T}$ is a wide κ-tree. For distinct nodes t and t', the branches b_t and $b_{t'}$ are eventually disjoint, since the functions f_t^G and $f_{t'}^G$ must be mutually generic. Thus $\dot{T}$ is normal. Also by density, every node has κ many successors at any level above. Whenever V_α is closed for both the label function ℓ and the width function wd, then $\dot{T}^G \cap V_\alpha$ is a subtree of $\dot{T}^G$ and $\dot{T}^G \cap V_\alpha \in V[G \cap V_\alpha]$. There are club many such α's. Furthermore, by a density argument, the ordinal $\mathsf{wd}(t)$ must be the least such that the branch below t is contained in the rectangle $\mathsf{wd}(t) \times \mathsf{ht}(t)$. We postpone the proof that $\dot{T}$ has no cofinal branches.

Item (5). Let $t \in T$ and $p \in \mathbb{Q}$. The set

$$\{t' \in T : p \Vdash t' <_{\dot{T}} t\}$$

is contained in the set $\mathsf{ran}(p_t)$, which has size $< \mu$. By the fact that T is $< \mu$-closed, it has a supremum. This set is linearly ordered, hence it has a unique supremum.

Item (6). Let p be a condition. We say that p satisfies $\boldsymbol{*(t)}$ if the domain of the function f_t^p is an ordinal and for any $t_0, t_1 \in \mathsf{ran}(f_t^p)$, p decides whether $t_0 <_{\dot{T}} t_1$ or $t_0 \not<_{\dot{T}} t_1$. It is straightforward to see that for any $A \subseteq T$, the set of conditions that satisfy $*(t)$ for every $t \in A$ is dense in $\mathbb{Q}$.

We show that if p satisfies $*(t)$, then it is nice with respect to t. Let $t \in T$ and let p satisfy $*(t)$. Let $\alpha < \kappa$ be such that $t \in T - V_\alpha$, and let $\bar{t} \in \mathsf{wd}(t) \times \mathsf{wd}(t)$. Let $j = \min(\kappa - \mathsf{dom}(f_t^p)$. Define $q \leqslant p$ by extending it only at coordinate t by letting

$$f_t^q := f_t^p \cup \{(j, \bar{t})\}.$$

Then clearly $q \cap V_\alpha = p \cap V_\alpha$, and q is minimal in the sense that $\dot{\pi}^q(t') = \dot{\pi}^p(t')$ when $t' \neq t$. Suppose that $q' \leqslant q$ is a condition that forces ``$\pi^p(t) <_{\dot{T}} \bar{t}$''. We claim that q' forces $\bar{t} <_{\dot{T}} t$. To see this, fix a generic $G \subseteq \mathbb{Q}$ that contains the condition q'. Then $\dot{\pi}^p(t)^G$ belongs to the branch below t, and also $\dot{\pi}^p(t)^G \leqslant \bar{t}$. By construction, there is a step i in the definition of the branch below t such that $t_i \leqslant \dot{\pi}^p(t)^G$ and j is the least index in $[i, \mu)$ such that $f_t^G(j)$ is a node above t_i. Since $f_t^G(j) = \bar{t}$, we will have $\bar{t} \in b_t$, i.e. $\bar{t} <_T t$. Thus q' forces $\bar{t}_{\dot{T}} t$, as wanted. This is enough to show that nodes that are nice with respect to nodes in A for any $A \subseteq T$ are dense in $\mathbb{Q}$.

Item (7) is clear.

$\dot{T}$ has no cofinal branches. We show that $\dot{T}$ is a name for a wide κ-Aronszajn tree. Suppose to the contrary that there is a condition $p \in \mathbb{Q}$ and a name $\dot{b}$ such that

$$p \Vdash \text{``}\dot{b} \text{ is a cofinal branch through } \dot{T}\text{''}.$$

Let θ be some large enough cardinal, let $M \preccurlyeq H_\theta$ be a model of size $< \kappa$ which is closed under the functions $t \mapsto \mathsf{wd}(t)$ and $t \mapsto \ell(t)$, such that $\alpha := \kappa \cap M$ is an ordinal and $p, \dot{b} \in M$. Find $q \leqslant p$ that decides the node $t := \dot{b}(\alpha)$ and belongs to the dense set from item (6). Then find $w \leqslant q \restriction \alpha$ in $\mathbb{Q} \cap M$ and two distinct nodes t^L and t^R in $\mathsf{wd}(t) \times \mathsf{ht}(t)$ at the some height $\bar{\alpha} < \alpha$ such that

$$w \Vdash \text{``}\pi^q(t) <_{\dot{T}} t^L, t^R\text{''}.$$

By item (6) there are two extensions $q^L, q^R \leqslant q$ such that $q^L \restriction M = q^R \restriction M = q \restriction M$ and such that any common extension of w and q^L (respectively q^R) forces ``$t^L <_{\dot{T}} t$'' (respectively ``$t^R <_{\dot{T}} t$''). Up to extending w further inside M, we may assume that it decides the node $\bar{t} := \dot{b}(\bar{\alpha})$. If $\bar{t} \neq t^L$, then $w \cup q^L$ is a condition that forces both $\bar{t} <_{\dot{T}} t$ and $t^L <_{\dot{T}} t$, which is a contradiction, and if $\bar{t} \neq t^R$, then $r \cup q^R$ is a condition that forces both $\bar{t} <_{\dot{T}} t$ and $t^R <_{\dot{T}} t$, which is also a contradiction. Hence $\dot{T}$ must be a wide κ-Aronszajn tree.

□

This ends the proof of the proposition.

□

Chapter 5

Trees at $\aleph_1$

In this chapter we show the consistency of the existence of a universal wide $\aleph_1$-Aronszajn tree, assuming the existence of a weakly compact cardinal. The proof consists of performing a finite support iteration $(\mathbb{P}_\delta : \delta \leqslant \kappa^+)$, where κ is a weakly compact cardinal. The first poset collapses the weakly compact onto $\aleph_1$ and introduces a wide κ-Aronszajn tree $\dot{T}$ as in Proposition 4.2.1, and the rest of the iteration takes care of embedding each wide κ-Aronszajn tree, also those that appear along the iteration, into $\dot{T}$. Preservation of κ will be guaranteed by strong properness with respect to enough models of size $< \kappa$ and the preservation of the Aronszajnness of $\dot{T}$ will be guaranteed using the reflection properties of the weakly compact.

We will use models from Definition 4.1.4 as side conditions. The conditions of the poset are will be finite and thus in some places the proof is substantially different from the proof of the analogous result for double successor cardinals. Therefore the analogous result for a double successor cardinal is presented in a separate chapter.

5.1 The embedding poset

Assumption. *For the rest of the chapter, we assume that* GCH *holds in* V. *We also fix a weakly compact cardinal* κ, *a* $\mathrm{Col}(\omega, <\kappa)$*-name for a wide* κ*-Aronszajn tree* $\dot{T}$ *and width and labeling functions* $t \mapsto \mathrm{wd}(t)$ *and* $t \mapsto \ell(t)$, *as in Proposition 4.2.1, with* $\mu = \omega$.

We next fix a bookkeeping function. We intentionally leave some flexibility regarding it, for instance, for now we will only assume that it picks wide trees, which are not necessarily assumed to be Aronszajn.

Notation 5.1.1. Fix a bookkeeping function

$$\kappa^+ \to \mathscr{P}(V_\kappa), \quad \gamma \mapsto \dot{S}_\gamma$$

such that whenever the poset $\mathbb{P}_\gamma$ has been defined, then $\dot{S}_\gamma$ is a $\mathbb{P}_\gamma$-name for a normal tree with domain $\kappa \times \kappa$ and that $\Vdash_{\mathbb{P}_\gamma} \mathsf{Lev}_\alpha(\dot{S}_\gamma) = \kappa \times \{\alpha\}$. We denote by S_γ the domain of $\dot{S}_\gamma$, i.e. the set $\kappa \times \kappa$.

Definition 5.1.2. Let S be a tree. A node $s \in S$ is an **exit node** from a set M if $s \notin M$ but $b_s \subseteq M$, i.e. the branch below s is contained in M.

We will now define the posets $\mathbb{P}_\delta$ by induction on $\delta \leqslant \kappa^+$. We follow the convention that $\mathbb{P}_0 = \{\varnothing\}$.

Notation 5.1.3. Assume that the posets $\mathbb{P}_\gamma$, $\gamma < \delta$, have already been defined for some $\delta \leqslant \kappa^+$. For each $\mathbb{P}_\gamma$, let $\mathcal{E}_\gamma$ and $(M^\gamma_\alpha : \alpha \in \mathcal{E}_\gamma)$ be as in Definition 4.1.4. We assume without loss of generality that each model M^γ_α contains the bookkeeping function.

The models $(M^\gamma_\alpha : \alpha \in \mathcal{E}_\gamma)$, $\gamma < \delta$, are the ones that will be used when defining the poset $\mathbb{P}_\delta$. When the poset $\mathbb{P}_\delta$ is defined, Definition 4.1.4 gives us the models $(M^\delta_\alpha : \alpha \in \mathcal{E}_\delta)$, which will again be used as side conditions when defining the poset $\mathbb{P}_{\delta+1}$, et cetera. We assume without loss of generality that each M^γ_α is closed for the fixed width and label maps $t \mapsto \mathsf{wd}(t)$ and $t \mapsto \ell(t)$.

For a tree S, we say that a subset $X \subseteq S$ is **closed under S-meets** if whenever $s, t \in X$, then also $s \wedge t \in X$. For a set of ordinals Y, we denote by $\lim Y$ the set of limit points of Y that are in Y.

Definition 5.1.4. Let $\delta \leqslant \kappa^+$. Conditions in the poset $\mathbb{P}_\delta$ are functions

$$p : \delta \to V_\kappa$$

such that for every $\gamma < \delta$,

$$p(\gamma) = (f^p_\gamma, N^p_\gamma)$$

satisfies

1. $f^p_0 \in \mathsf{Col}(\omega, <\kappa)$,
2. for non-zero $\gamma < \delta$, $f^p_\gamma : \kappa \times \kappa \to \kappa \times \kappa$ is a finite partial injective map such that:

   (a) $p \restriction \gamma$ decides the $\dot{S}_\gamma$-meets in the set $\mathsf{dom}(f^p_\gamma)$,
   (b) $p \restriction \gamma \Vdash$ ``$\mathsf{dom}(f^p_\gamma)$ is closed under $\dot{S}_\gamma$-meets",
   (c) $p \restriction \gamma \Vdash$ ``$f^p_\gamma : \dot{S}_\gamma \to \dot{T}$ is a level- and meet-preserving tree-embedding",

3. $N^p_\gamma \subseteq \lim \mathcal{E}_\gamma$ is a finite set such that whenever $\alpha \in N^p_\gamma$ and $\xi \in \gamma \cap M^\gamma_\alpha$, then $\alpha \in N^p_\xi$, and such that the union

$$\bigcup_{\gamma<\delta} N^p_\gamma$$

   is finite,

4. for every non-zero $\gamma < \delta$, every $s \in \mathsf{dom}(f^p_\gamma)$ and $\alpha \in N^p_\gamma$:
   (a) $s \in M^\gamma_\alpha$ if and only if $f^p_\gamma(s) \in M^\gamma_\alpha$,
   (b) $p \restriction \gamma \Vdash$ ``s is an exit node from M^γ_α if and only if $f^p_\gamma(s)$ is",
   (c) if $p \restriction \gamma \Vdash$ ``s is an exit node from M^γ_α", then $p \restriction \gamma$ decides the ordinal

$$\mathsf{wd}(s) := \text{the least } \beta \text{ in } \lim \mathcal{E}_\gamma - \mathsf{ht}(s) \text{ such that } b_s \subseteq \beta \times \mathsf{ht}(s),$$

   and it satisfies $\mathsf{wd}(s) = \mathsf{wd}(f^p_\delta(s))$,
   (d) if $s \notin M^\gamma_\alpha$, then there is $\bar{s} \in \mathsf{dom}(f^p_\gamma) - M^\gamma_\alpha$ such that

$$p \restriction \gamma \Vdash \text{``}\bar{s} \leqslant_{\dot{S}_\gamma} s \text{ and } \bar{s} \text{ is an exit node from } M^\gamma_\alpha\text{"},$$

   (e) if $p \restriction \gamma \Vdash$ ``s is an exit node from M^γ_α", then the label $\ell(f^p_\gamma(s))$ satisfies:
      i. $\ell(f^p_\gamma(s)) \in \mathcal{E}_\gamma$,
      ii. $s \in M^\gamma_{\ell(f^p_\gamma(s))}$,
      iii. $\ell(f^p_\gamma(s)) \in N^p_\xi$ for every $\xi \in \gamma \cap M^\gamma_{\ell(f^p_\gamma(s))}$,

5. the support $\mathsf{sp}(p) := \{\gamma \in \delta : f^p_\gamma \neq \varnothing\}$ is finite.

The order is the pointwise inverse inclusion: $q \leqslant p$ if $f^q_\gamma \supseteq f^p_\gamma$ and $N^q_\gamma \supseteq N^p_\gamma$ for all $\gamma < \delta$.

Remark 5.1.5. The set $\{\gamma < \delta : N^p_\gamma \neq \varnothing\}$ is not necessarily finite, even if the union $\bigcup_{\gamma<\delta} N^p_\gamma$ is.

Remark 5.1.6. A word about item (4e) of the definition of the poset. If ρ is the label of t and $\rho \in \mathcal{E}_\delta$, then $t \notin M^\delta_\rho$ and t is an exit node from M^δ_ρ. In particular, if $s \in \mathsf{dom}(f^p_\delta)$ is an exit node from M^δ_α for some $\alpha \in N^p_\delta$, and $f^p_\delta(s) = t$, then the model $M^\delta_{\ell(t)}$ "separates" the nodes s and t in the sense that

$$s \in M^\delta_{\ell(t)} \text{ and } t \notin M^\delta_{\ell(t)}.$$

In particular, it cannot happen that $\ell(t) \in N^p_\delta$, since each model with index in N^p_δ must be closed under the function f^p_δ. However, the model $M^\delta_{\ell(t)}$ must appear in the side conditions of the previous coordinates, i.e. it must hold that $\ell(t) \in N^p_\gamma$ for every $\gamma \in \delta \cap M^\delta_{\ell(t)}$.

Lemma 5.1.7. *If $\gamma < \delta$ and $q \in \mathbb{P}_\gamma$ extends $p \restriction \gamma$, then the concatenation $q^\frown p \restriction [\gamma, \delta)$ is a condition in $\mathbb{P}_\delta$ extending q and p. In particular $\mathbb{P}_\gamma \subseteq_c \mathbb{P}_\delta$.*

Proof. Straightforward verification. □

Notation 5.1.8. Each $\mathbb{P}_\delta$, $\delta < \kappa^+$, has size κ and can thus be coded as a subset of V_κ using any injection $\delta \to \kappa$. We tacitly identify each $\mathbb{P}_\delta$ with an isomorphic poset contained in V_κ, and assume that this identification was done using a coding function belonging to each model M^δ_α, $\alpha < \kappa$.

Remark 5.1.9. The final poset $\mathbb{P}_{\kappa^+}$ is the direct limit of the posets $(\mathbb{P}_\delta : \delta < \kappa^+)$, via the maps $p \mapsto p^\frown(\varnothing, \varnothing), \dots$. In particular, since each $\mathbb{P}_\delta$, $\delta < \kappa^+$, has size κ, it follows that $\mathbb{P}_{\kappa^+}$ has κ^+-cc.

The iteration $(\mathbb{P}_\delta : \delta \leqslant \kappa^+)$ has now been defined. It will be seen that it has the following properties:

1. $\mathbb{P}_{\kappa^+}$ has κ^+-cc, so it preserves all cardinals $\lambda \geqslant \kappa^+$,
2. $\mathbb{P}_{\kappa^+}$ collapses every $\alpha < \kappa$ to be countable,
3. $\mathbb{P}_{\kappa^+}$ preserves κ and adds κ^+ many reals,
4. there is an injective level-preserving tree-embedding $\dot{f}_\gamma : \dot{S}_\gamma \to \dot{T}$ in $V^{\mathbb{P}_{\kappa^+}}$, for every $\gamma < \kappa^+$,
5. if the bookkeeping function picks only names for wide κ-Aronszajn trees, then $\dot{T}$ is a wide $\aleph_1$-Aronszajn tree in $V^{\mathbb{P}_{\kappa^+}}$.

In particular, with a suitable bookkeeping function, $\dot{T}$ will be a universal wide $\aleph_1$-Aronszajn tree in $V^{\mathbb{P}_{\kappa^+}}$.

The first two properties are immediate from Remark 5.1.9 and the definition of the poset. The rest of the paper is devoted to proving the remaining three properties.

5.2 Properties of the poset

In this section, it will be shown that $\mathbb{P}_\delta$, for each $\delta < \kappa$, is strongly proper with respect to stationarily many $M \in [H_\lambda]^{<\kappa}$, for any large enough regular cardinal λ, and furthermore, that for every $\gamma < \kappa^+$ and $s \in S_\gamma$, the set of conditions p with $s \in \mathsf{dom}(f^p_\gamma)$ is dense. This will imply that $\mathbb{P}_{\kappa^+}$ preserves κ, making it the new $\aleph_1$, that $\mathbb{P}_{\kappa^+}$ adds κ^+ many reals, making continuum to be of size $\aleph_2$, and that $\dot{T}$ is made universal for wide κ-Aronszajn trees. The section after this one is dedicated to showing that $\dot{T}$ itself remains (wide) Aronszajn.

We now decribe informally the simple idea behind the proof of strong properness. First, note why it is important that the tree $\dot{T}$ is not in V, but rather introduced by the first poset $\mathbb{P}_1$. Indeed, suppose for the sake of argument that we tried to add a tree-embedding

$$\dot{f} : S \to T$$

using finite approximations $p : S \to T$ of a tree-embedding as conditions, from a (wide) $\aleph_1$-tree S into a tree T both of which are in V. The ordering is inverse inclusion. Suppose we now want to show that this poset is strongly proper. Fix some countable model M such that $S, T \in M$. We argue that no condition p can have a residue into M. Indeed, fix some condition p which is strong enough that it contains some nodes in S that are in $S - M$ and suppose that it had a residue r into M. Now choose a node $s \in S - M$ at height $\omega_1 \cap M$ whose predecessors are all in M. Look at the branch below s - there is a maximal node $\bar{s}$ which is both below s and in the domain of r. Choose some node s' which is above $\bar{s}$ and below s and another node t' in $T \cap M$ which is above $r(\bar{s})$ but *not* below $p(s)$. Extend the condition r by letting $w := r \cup \{(s', t')\}$. Then w extends r and belongs to M. However, this w cannot be compatible with the original condition p since a tree-embedding must be order-preserving. Any common extension of w and p would map the node s' which is below s to the node t' which is not below $p(s)$.

We solve this problem by taking the tree T to be generic over V. Indeed, suppose now that our conditions are pairs $p = (u^p, f^p)$, where u^p is a finite approximation of a wide $\aleph_1$-tree $\dot{T}$ and u^p forces that f^p is a finite approximation of a tree-embedding

$$f^p : S \to \dot{T},$$

where S is some $\aleph_1$-tree in V. We show that the above problem does not occur. Suppose for simplicity that we are in the following situation: p is a condition and M is a model such that the model M is closed under the finite tree-embedding f^p, i.e. a node $s \in \mathsf{dom}(f^p)$ is in M if and only if its image $f^p(s)$ is in M (in our case, this situation will be guaranteed by the usage of side conditions). Now consider the trace $p \cap M := (u^p \cap M, f^p \cap M)$. This is still a condition. For every node $t \in \mathsf{ran}(f^p) - M$, there is a node $\pi(t)$ which is forced by u^p to be the maximal node which is both below t and in M. Namely, this $\pi(t)$ is the node in $u^p \cap M$ which is below t in the order $<_{u^p}$ and at a maximal height. Now, we extend the trace $p \cap M$ adding a preimage for the node $\pi(t)$. Whenever $(s,t) \in f^p - M$, we look at the predecessor of s at the height if $\pi(t)$ and call it $\bar{s}$. We then add the pairs $(\bar{s}, \pi(t))$ to the trace. In other words, we define

$$r := (p \cap M) \cup \{(\bar{s}, \pi(t)) : (s,t) \in f^p - M\}.$$

It can be shown that r is a residue of p into M. For instance, the previous problem cannot arise: suppose for simplicity that w extends r just by one pair (s', t') and suppose that s' is below some node $s \in \mathsf{dom}(f^p) - M$. But now the condition p does not decide whether t' is below $f^p(s)$ or not. Therefore, we are free to take the pointwise union of w and p and extend its tree-part $u^w \cup u^p$ to some u which decides that t' is indeed below $f^p(s)$. If it happened that s' was not below s, we would have done the same, except extended the tree part $u^w \cup u^p$ to some u that decides that t' is not below $f^p(s)$. This is the idea in the proof of strong properness. The difference is that we need to take care of trees $\dot{S}$ that are not necessarily in V, and we have multiple embeddings to deal with. This idea will be exploited in what follows.

The following notation will be used throughout the rest of the chapter.

Definition 5.2.1 (E^p_δ). Let $\delta < \kappa^+$ and $p \in \mathbb{P}_\delta$. Define

$$\boldsymbol{E^p_\delta} := \{\alpha \in \mathcal{E}_\delta : \alpha \text{ belongs to } N^p_\gamma \text{ for every } \gamma \in \delta \cap M^\delta_\alpha\}$$

Strong properness of $\mathbb{P}_\delta$ with respect to a model M^δ_α is shown in two steps:

1. if $p \in \mathbb{P}_\delta \cap M^\delta_\alpha$, then there is $q \leqslant p$ such that $\alpha \in E^q_\delta$,
2. if $\alpha \in E^q_\delta$, then q has a residue into M^δ_α.

In particular, if $\alpha \in E^q_\delta$, then q is strongly $(\mathbb{P}_\delta, M^\delta_\alpha)$-generic. This will imply that each poset $\mathbb{P}_\delta$, $\delta < \kappa^+$, is strongly proper with respect to the models

$$(M^\delta_\alpha : \alpha \in \mathcal{E}_\delta).$$

The first step is easy. The second step is hard.

Lemma 5.2.2. *Let* $\delta < \kappa^+$ *and* $\alpha \in \mathcal{E}_\delta$. *If* $p \in \mathbb{P}_\delta \cap M^\delta_\alpha$, *then the condition* $q \leqslant p$ *defined by*

$$q(\gamma) := \begin{cases} (f^p_\gamma, N^p_\gamma \cup \{\alpha\}), & \text{if } \gamma \in \delta \cap M^\delta_\alpha, \\ (\varnothing, \varnothing) & \text{otherwise.} \end{cases}$$

is an extension of p *such that* $\alpha \in E^q_\gamma$.

Proof. It is clear that $\alpha \in E^q_\delta$. The fact that q extends p follows from the fact that the support of p is contained in the model M^δ_α. It is straightforward to verify that q is a condition: all of the clauses of the definition of the poset follow from the fact that for every $\gamma \in \delta \cap M^\delta_\alpha$, the function f^q_γ belongs to the model M^γ_α. And since each function f^q_γ belongs to M^γ_α, there are no exit nodes from M^γ_α in $\mathsf{dom}(f^q_\gamma)$, so the clauses in the definition of the poset $\mathbb{P}_\delta$ regarding exit nodes can be ignored. □

That was the first step. We will first prove a node density claim and come back to the second step in the subsection after.

5.2.1 Node density

In this subsection we will prove a version of a density lemma that will be used in the proof of strong properness. We begin with a simple lemma that allows to add nodes to the embedding approximation part of a condition, given that they are forced to be below nodes already in the embedding approximation.

Lemma 5.2.3. *Let* $\delta \leqslant \kappa^+$, $p \in \mathbb{P}_\delta$ *and* $\gamma < \delta$. *Assume that* $(s, t) \in f^p_\gamma$ *and* $(\bar{s}, \bar{t})$ *is a pair of nodes that satisfies:*

- $\bar{s} \in S_\gamma$, $\bar{t} \in T$ *and* $\mathsf{ht}(\bar{s}) = \mathsf{ht}(\bar{t})$,
- $p \restriction \gamma \Vdash$ "$\bar{s} <_{\dot{S}_\gamma} s$ *and* $\bar{t} <_{\dot{T}} t$".

Then q *obtained from* p *by letting*

$$f^q_\gamma := f^p_\gamma \cup \{(\bar{s}, \bar{t})\},$$

$N^q_\gamma := N^p_\gamma$ *and* $q(\xi) := p(\xi)$ *for* $\xi \in \delta - \{\gamma\}$, *is a condition in* $\mathbb{P}_\delta$ *that satisfies* $q \leqslant p$.

Proof. Straightforward verification of the items in Definition 5.1.4. □

The following node density lemma is conditioned on the strong properness of $\mathbb{P}_\delta$. The lemma will be used in the proof of strong properness of $\mathbb{P}_{\delta+1}$.

Lemma 5.2.4 (Node Density)**.** *Let $\delta < \kappa^+$. Assume that it holds for every condition $p \in \mathbb{P}_\delta$ that if $\alpha \in E^p_\delta$, then p has a residue into M^δ_α. Then, for every $p \in \mathbb{P}_{\delta+1}$ and $s \in S_\delta$ there is $q \leqslant p$ such that $s \in \mathsf{dom}(f^q_\gamma)$. Moreover, the extension $q \leqslant p$ can be chosen to be minimal in the sense that $N^q_\delta = N^p_\delta$ and q forces that $\mathsf{dom}(f^q_\delta)$ is the least set containing $\mathsf{dom}(f^p_\delta) \cup \{s\}$ and being closed under $\dot{S}_\delta$-meets and exit nodes from M^δ_α for $\alpha \in N^q_\delta$.*

Proof. Let $p \in \mathbb{P}_{\delta+1}$ and $s \in S_\delta$. Up to extending $p \restriction \delta$, we may assume that it decides the non-trivial meets in $\mathsf{dom}(f^p_\delta) \cup \{s\}$. In fact, there is at most one meet $s \wedge s'$ not already in the domain of f^p_δ. We may also assume that $p \restriction \delta$ decides its implicit image

$$t_{s' \wedge s} := \text{the unique node below } f^p_\delta(s') \text{ at the height of } s' \wedge s,$$

where $s' \in \mathsf{dom}(f^p_\delta)$ is any node such that $s \wedge s' \notin \mathsf{dom}(f^p_\delta)$. It is straightforward to verify that

$$p \restriction \delta^\frown(f^p_\delta \cup \{(s' \wedge s, t_{s' \wedge s})\}, N^p_\delta)$$

is a condition in $\mathbb{P}_{\delta+1}$ extending p. Therefore from now onwards, we will assume without loss of generality that $s' \wedge s \in \mathsf{dom}(f^p_\delta)$ and only worry about s itself and exit nodes below s from models M^δ_α, where $\alpha \in N^p_\delta$.

Up to extending $p \restriction \delta$ further, we assume that $p \restriction \delta$ decides the width $\mathsf{wd}(s)$ of s as well as for every $\alpha \in N^p_\delta$ such that $s \notin M^\delta_\alpha$, $p \restriction \delta$ decides the node

$$s_\alpha := \text{the unique exit node from } M^\delta_\alpha \text{ below } s,$$

as well as the widths $\mathsf{wd}(s_\alpha)$. Note that it is possible that $s_\alpha = s_\beta$ for $\alpha < \beta$ from N^p_δ. The difficulty in adding each node s_α into $\mathsf{dom}(f^p_\delta)$ is that we need to find an image t_α for it which has a label $\ell(t)$ that separates s_α and t_α and can be added to E^p_δ. Enumerate the set N^p_δ in an increasing order as

$$\alpha_0 < \cdots < \alpha_{n-1}.$$

Let $\alpha_n := \kappa$. Note that by assumption $p \restriction \delta$ has a residue to each $M^\delta_{\alpha_k}$, because $N^p_\delta \subseteq E^{p \restriction \delta}_\delta$. Let $r_n := p \restriction \delta$ and by reverse recursion on $k < n$, choose r_k such that

$$r_k \text{ is a residue of } r_{k+1} \text{ into } M^\delta_{\alpha_k}.$$

We will work with this sequence of conditions $(r_k)_{k\leqslant n}$. By (non-reverse) recursion on $k \leqslant n$, we build functions f_k and conditions $v_k \in \mathbb{P}_\delta \cap M^\delta_{\alpha_k}$.

Step 0: Let $f_0 := f^p_\delta$ and $v_0 := r_0$.

Step $k+1$: Assume that f_k and $v_k \in \mathbb{P}_\delta \cap M^\delta_{\alpha_k}$ were defined and satisfy $v_k \leqslant r_k$. We define f_{k+1} and v_{k+1}. By assumption $v_k \in \mathbb{P}_\delta \cap M^\delta_{\alpha_k}$ and $v_k \leqslant r_k$. Note that also $r_k, r_{k+1}, \mathbb{P}_\delta, M^\delta_{\alpha_k} \in M^\delta_{\alpha_{k+1}}$. Thus

$$M^\delta_{\alpha_{k+1}} \models r_k \text{ is a residue of } r_{k+1} \text{ into } M^\delta_{\alpha_k}.$$

Hence, we may choose $v' \in \mathbb{P}_\delta \cap M^\delta_{\alpha_{k+1}}$ such that

$$v' \leqslant v_k, r_{k+1}.$$

If $s_{\alpha_k} \notin M^\delta_{\alpha_{k+1}}$, then let

$$f_{k+1} := f_k \text{ and } v_{k+1} := v'$$

and move to step $k+2$. Otherwise $s_{\alpha_k} \in M^\delta_{\alpha_{k+1}}$. We will extend v' further and find $f_{k+1} \supseteq f_k$ such that

$$\mathsf{dom}(f_{k+1}) = \mathsf{dom}(f_k) \cup \{s_{\alpha_k}\}.$$

Since v' is an element in $M^\delta_{\alpha_{k+1}}$ and α_{k+1} is a limit point of $\mathcal{E}_\delta$, there must be $\beta_k \in \mathcal{E}_\delta \cap M^\delta_{\alpha_{k+1}}$ such that

$$v' \in M^\delta_{\beta_k}.$$

This β_k will be the label of the image of s_{α_k}, if $s_{\alpha_k} \in M^\delta_{\alpha_{k+1}}$. Now $\alpha_k, \alpha_{k+1} \in \lim \mathcal{E}_\delta$, $\beta_k \in \mathcal{E}_\delta$ and

$$\alpha_k < \beta_k < \alpha_{k+1}.$$

By Lemma 5.2.2 applied inside the model $M^\delta_{\alpha_{i+1}}$, v'' defined by

$$v''(\gamma) = \begin{cases} (f^{v'}_\gamma, N^{v'}_\gamma \cup \{\beta_k\}), & \text{if } \gamma \in M^\delta_{\beta_k}, \\ (\varnothing, \varnothing), & \text{otherwise} \end{cases}$$

is a condition in $\mathbb{P}_\delta$, in particular satisfying $\beta_k \in E^{v''}_\delta$. Furthermore, $v'' \in \mathbb{P}_\delta \cap M^\delta_{\alpha_{k+1}}$ and

$$v'' \leqslant v' \leqslant r_{k+1}.$$

Consider the node s_{α_k}. It has width $\mathsf{wd}(s_{\alpha_k}) \leqslant \alpha_k$ because it is an exit node from $M^\delta_{\alpha_k}$. Choose a node $t_{\alpha_k} \in T$ such that $\mathsf{ht}(t_{\alpha_k}) = \mathsf{ht}(s_{\alpha_k})$, $\mathsf{wd}(t_{\alpha_k}) = \mathsf{wd}(s_{\alpha_k})$ and $\ell(t) = \beta_k$. By elementarity we may assume $t_{\alpha_k} \in M^\delta_{\alpha_{k+1}}$. Then t_{α_k} is forced to be an exit node from $M^\delta_{\alpha_k}$. Furthermore, the condition v'' does not decide anything about t_{α_k}, because t_{α_k} was chosen outside of $M^\delta_{\beta_k}$, so by extending the collapse part $f_0^{v''}$, we obtain a condition $v_{k+1} \leqslant v''$, still in $\mathbb{P}_\delta \cap M^\delta_{\alpha_{k+1}}$, that forces for every $j \leqslant k$, $t_{s' \wedge s} \leqslant_{\dot{T}} t_{\alpha_j} <_{\dot{T}} t_{\alpha_k}$, and for every $s' \in \mathsf{dom}(f^p_\delta) \cap M^\delta_{\alpha_{k+1}}$

$$f^p_\delta(s') \wedge t_{\alpha_k} = t_{s' \wedge s}.$$

Define

$$f_{k+1} := f_k \cup \{(s_{\alpha_k}, t_{\alpha_k})\}.$$

If $s \in M^\delta_{\alpha_{k+1}}$, then choose also a node $t_s \in (T \cap M^\delta_{\alpha_{k+1}}) - \mathsf{ran}(f^p_\delta)$ such that $\mathsf{ht}(t_s) = \mathsf{ht}(s)$ and $t_{\alpha_0}, \ldots, t_{\alpha_{n-1}} \in \mathsf{wd}(t_s) \times \mathsf{ht}(t_s)$. We may assume that it is fresh in the sense that v_{k+1} does not decide anything about it. Up to extending v_{k+1} once more at the collapse coordinate, we may assume that it forces

$$t_{\alpha_k} \leqslant_{\dot{T}} t_s.$$

In this case, extend f_{k+1} also by the pair (s, t_s).

This ends the recursion. We have $v_n \leqslant p \restriction \delta$ and

$$v_n \Vdash ``f_n \text{ is level- and meet-preserving tree-embedding}''.$$

Now $f_n \supseteq f^p_\delta$ and $s \in \mathsf{dom}(f_n)$. Moreover,

$$q := v_n{}^\frown(f_n, N^p_\delta)$$

is a condition that extends p, as wanted. This concludes the proof of Node Density Lemma 5.2.4. ☐

5.2.2 Strong Properness

As was mentioned above, the strong properness of $\mathbb{P}_\delta$ with respect to a model M^δ_α is shown by showing the two following statements:

1. If $p \in \mathbb{P}_\delta \cap M^\delta_\alpha$, then there is $q \leqslant p$ such that $\alpha \in E^q_\delta$.
2. If $\alpha \in E^q_\delta$, then q has a residue into M^δ_α.

The first step was shown in Lemma 5.2.2. We collect preliminary definitions and lemmas before presenting the proof of the second step.

Definition 5.2.5. Let $\delta < \kappa^+$. The **trace** of a condition $p \in \mathbb{P}_\delta$ to a model M is defined to be the function $[p]_M$ on δ such that

$$[p]_M(\gamma) = \begin{cases} (f^p_\gamma \cap M, N^p_\gamma \cap M) & \text{if } \gamma \in \delta \cap M, \\ \varnothing & \text{if } \gamma \in \delta - M. \end{cases}$$

The trace of a condition is not necessarily a condition. Residues of conditions will be extensions of their traces. We write $q \leqslant [p]_M$ if $f^q_\gamma \supseteq f^p_\gamma \cap M$ and $N^q_\gamma \supseteq N^p_\gamma \cap M$ for all $\gamma \in \delta \cap M$, even if the trace $[p]_M$ was not a condition.

Notation 5.2.6. For a finite set $E \subseteq \kappa^+ \times \kappa$ of pairs of ordinals we write

$$E_\gamma := \{\beta : (\gamma, \beta) \in E\}$$

and when the set E is clear from context, we also denote by $\boldsymbol{(\gamma, \beta^+)}$ the pair where

$$\beta^+ := \begin{cases} \text{the least element of } E_\gamma \text{ strictly above } \beta, & \text{if } \beta < \max(E_\gamma), \\ \kappa, & \text{if } \beta = \max(E_\gamma), \end{cases}$$

for any $\beta < \kappa$. Furthermore, we denote

$$E \restriction \delta := E \cap (\delta \times \kappa).$$

Recall from item (6) of Proposition 4.2.1 that a condition $f \in \mathsf{Col}(\omega, <\kappa)$ is **nice with respect to a node $\boldsymbol{t}$** if for any α with $t \notin V_\alpha$, any $w \in \mathsf{Col}(\omega, <\kappa) \cap V_\alpha$ and any $\bar{t} \in \mathsf{wd}(t) \times \mathsf{ht}(t)$, if $w \leqslant p \cap V_\alpha$ and $w \Vdash$ "$\pi^p(t) <_{\dot{T}} \bar{t}$", then there is $q \leqslant p$ such that $q \cap v_\alpha = p \cap V_\alpha$, q is minimal in the sense that $\pi^q(t') = \pi^p(t')$ for any $t' \neq t$, and that any common extension of w and q forces $\bar{t} <_{\dot{T}} t$. Recall also that the set of conditions that are nice with respect to t for every $t \in A$ for any $A \subseteq T$, are dense in $\mathsf{Col}(\omega, <\kappa)$.

Definition 5.2.7. Let $\delta < \kappa^+$, $p \in \mathbb{P}_\delta$ and $\alpha \in E^p_\delta$. A **residue system for $\boldsymbol{p}$ into $\boldsymbol{M^\delta_\alpha}$** is a tuple

$$\vec{r}_E = (r_{(\gamma,\beta)} : (\gamma, \beta) \in E)$$

indexed by a finite set $E \subseteq \kappa^+ \times \kappa$ such that if we denote

$$r_{(\gamma,\kappa)} := p \restriction \gamma,$$

then the following are satisfied:

1. $r_{(\gamma,\beta)} \in \mathbb{P}_\gamma \cap M^\gamma_\beta$ for every $(\gamma, \beta) \in E$.

2. E is the least set such that $E_\delta = \{\alpha\}$ and E is **closed** in the following sense: for every $(\gamma, \beta) \in E$ and $\xi \in \mathsf{sp}(r_{(\gamma,\beta+)}) \cap M^\gamma_\beta$ with $\xi \geqslant 1$,
$$E^{r_{(\gamma,\beta+)}}_\xi - \beta \subseteq E_\xi.$$
Furthermore, if $\beta^+ \neq \kappa$, then $\beta^+ \in E_\xi$.
3. Let $(\gamma, \beta), (\gamma', \beta') \in E$. If $\beta < \beta'$, or if $\beta = \beta'$ and $\gamma > \gamma'$, then
$$r_{(\gamma,\beta)} \restriction \min\{\gamma, \gamma'\} \leqslant [r_{(\gamma',\beta')} \restriction \min\{\gamma, \gamma'\}]_{M^\gamma_\beta}$$
In particular, $\bigcup_{x\in E} f^{r_x}_0$ is a condition in $\mathsf{Col}(\omega, < \kappa)$.
4. For every $(\gamma, \beta) \in E$ and every non-zero $\xi \in \gamma \cap M^\gamma_\beta$ and for every pair $(s, t) \in f^{r_{(\gamma,\beta+)}}_\xi$ of exit nodes from M^ξ_β, if ρ is the maximal ordinal in E_ξ such that $t \notin M^\xi_\rho$, then $r_{(\xi,\rho)}$ decides the "implicit preimage" for the projection of the image of s, i.e. the node
$$\bar{s} := \text{the predecessor of } s \text{ at the height of } \pi^E(t),$$
where $\pi^E(t) := \pi^f(t)$ for $f = f^p_0 \cup \bigcup_{x\in E} f^{r_x}_0$.
5. For every $(\gamma, \beta) \in E$ and non-zero $\xi \in \gamma \cap M^\gamma_\beta$, the collapse part $f^{r_{(\gamma,\beta+)}}_0$ is nice with respect to t for every $t \in \mathsf{ran}(f^{r_{(\gamma,\beta+)}}_\xi)$ that is an exit node from M^γ_β.

We call the condition $r_{(\delta,\alpha)}$ the **root condition** of the system $\vec{r}_E$.

Notation 5.2.8. If $\vec{r}_E$ is a residue system for $p \in \mathbb{P}_\delta$ into M^δ_α and there is no danger of confusion of p, we denote
$$r_{(\gamma,\kappa)} := p \restriction \gamma,$$
for every $\gamma < \delta$.

Lemma 5.2.9. *If $\vec{r}_E$ is a residue system for p into M^δ_α, then $f^p_0 \cup \bigcup_{x\in E} f^{r_x}_0$ is a condition in* $\mathsf{Col}(\omega, < \kappa)$.

Proof. Follows from item (3) of definition of residue system. □

Lemma 5.2.10. *If $\vec{r}_E$ is a residue system for p into M^δ_α and $r \in \mathbb{P}_\delta \cap M^\delta_\alpha$ extends $r_{(\delta,\alpha)}$, then the system obtained from $\vec{r}_E$ by replacing $r_{(\delta,\alpha)}$ by r is a residue system for p into M^δ_α.*

Recall the definition of path from 4.1.8.

Definition 5.2.11. If $E \subseteq \kappa^+ \times (\kappa+1)$ is a set of pairs, then a **path in $\boldsymbol{E}$** is a path whose each member belongs to E.

Lemma 5.2.12. *If $\vec{r}_E$ is a residue system for a condition $p \in \mathbb{P}_\delta$ into M^δ_α and $(\gamma, \beta) \in E$, then there is a path from M^δ_α to M^γ_β.*

Proof. Follows from the leastness of the set E. □

The following lemma is a technical lemma that allows to break the index set of a residue system into pairwise disjoint parts using the models that appear in it.

Lemma 5.2.13. *If $\vec{r}_E$ is a residue system for p into M^δ_α and $\gamma = \max(\mathrm{sp}(p) \cap M^\delta_\alpha)$, then $E - \{(\delta, \alpha)\}$ can be written as the disjoint union of the sets*

$$E^\beta := E \cap (M^\gamma_{\beta+} - M^\gamma_\beta)$$

where $\beta \in E_\gamma$, and furthermore, $\vec{r}_{E^\beta}$ is a residue system in $M^\gamma_{\beta+}$ for $r_{(\gamma,\beta^+)}$ into M^γ_β.

Proof. Enumerate the set E_γ as $\alpha = \beta_0 < \cdots < \beta_{n-1}$ and let $\beta_n := \kappa$ and denote $E^k := E \cap (M^\gamma_{\beta_{k+1}} - M^\gamma_{\beta_k})$. Fix some $k \leqslant n$. Then $\vec{r}_{E^k} \in M^\gamma_{\beta_{k+1}}$, since if $(\xi, \rho) \in E^k$, then $r_{(\xi,\rho)} \in M^\xi_\rho \subseteq M^\gamma_{\beta_{k+1}}$. We claim that $\vec{r}_{E^k}$ is a residue system for $r_{k+1} := r_{(\gamma,\beta_{k+1})}$ into $M^\gamma_{\beta_k}$.

We say that a path $(\gamma_k, \beta_k)_{k \leqslant m}$ is a **path in $\vec{\boldsymbol{r}}_{\boldsymbol{E}}$** if it is a path in E that satisfies additionally: $\gamma_{k+1} \in \mathrm{sp}(r_{(\gamma_k,\beta_k^+)}) \cap M^{\gamma_k}_{\beta_k}$ and $\beta_{k+1} \in E^{r_{(\gamma_k,\beta_k^+)}}_{\gamma_{k+1}} - \beta_k$ for every $k \leqslant m$.

Claim 5.2.14. *If $(\xi, \rho) \in E^k$, then $r_{(\xi,\rho^+)} \in M^\gamma_{\beta_{k+1}}$.*

Proof of Claim 5.2.14. The claim is vacuously true if $\xi = \gamma$, since then $\rho = \beta_k$ and $r_{(\gamma,\beta_k)} \in M^\gamma_{\beta_k} \subseteq M^\gamma_{\beta_k}$. We show by induction on the length of the path that if $(\xi_i, \rho_i)_{i \leqslant m}$ is a path in $\vec{r}_E$ with $\rho_m < \beta_{k+1}$, then $\rho^+_m \leqslant \beta_{k+1}$. Fix a path $(\xi_i, \rho_i)_{i \leqslant m+1}$ in $\vec{r}_E$ such that $\rho_{m+1} < \beta_{k+1}$. We claim that $\rho^+_{m+1} \leqslant \beta_{k+1}$. Now $\rho_m \leqslant \rho_{m+1}$, so by the induction hypothesis $\rho^+_m \leqslant \beta_{k+1}$. In paticular $\rho^+_m < \kappa$, so

$$\rho^+_m \in E_{\xi_{m+1}}.$$

But note that $\rho_{m+1} \in (E^{r_{(\xi_m,\rho^+_m)}}_{\xi_{m+1}} - \rho_m) \subseteq M^{\xi_m}_{\rho^+_m}$, so $\rho_{m+1} < \rho^+_m$. This implies that

$$\rho^+_{m+1} \leqslant \rho^+_m \leqslant \beta_{k+1}.$$

Thus, if $(\xi, \rho) \in E^k$, then there is path from M_α^δ to M_ρ^ξ in $\vec{r}_E$, and so $\rho < \beta_{k+1}$ implies $\rho^+ \leqslant \beta_{k+1}$, which in turn implies $r_{(\xi,\rho^+)} \in M_{\beta_{k+1}}^\gamma$. $\square$

Now, the above claim implies that $\vec{r}_{E^k}$ satisfies item (2) of the definition of residue system, namely, the following: E^k is the least set such that $E_\gamma^k = \{\beta_k\}$ and it is closed for the following: if $(\xi, \rho) \in E^k$ and $\xi' \in \mathsf{sp}(r_{(\xi,\rho^+)}) \cap M_\rho^\xi$, then

$$E_\xi^{r_{(\xi,\rho^+)}} - \rho \subseteq E_\xi^k,$$

and if $\rho^+ < \kappa$, then also $\rho^+ \in E_\xi^k$. It also also implies that if $(\xi, \rho) \in E^k$, then there is a path in $\vec{r}_{E^k}$ from $M_{\beta_k}^\gamma$ to M_ρ^ξ, and furthermore, that if $(\xi, \rho) \in E^{k+1} - E^k$, then $\rho \geqslant \beta_{k+1}$.

It remains to be shown that items (4) and (5) are satisfied.

Claim 5.2.15. *If $t \in T \cap (M_{\beta_{k+1}}^\gamma - M_{\beta_k}^\gamma)$, then $\pi^E(t) = \pi^{E^k}(t)$.*

Proof of Claim 5.2.15. Follows from the fact that the branch below t is decided by the poset $\mathsf{Col}(\omega, <\kappa) \cap (M_{\beta_{k+1}}^\gamma - M_{\beta_k}^\gamma)$, and from the fact that if $(\xi, \rho) \in E^{k+1} - E^k$, then $\rho > \beta_{k+1}$, and thus for any $(\xi', \rho') \in E^k$, the collapse conditions extend: $f_0^{r(\xi',\rho')} \leqslant [f_0^{r(\xi,\rho)}]_{V_{\rho'}}$. $\square$

Claim 5.2.15 implies that items (4) and (5) are satisfied. This concludes the proof of the lemma.

$\square$

We will now prove a lemma that is an analogue of the flexibility property of the tree $\dot{T}$ in $\mathsf{Col}(\omega, <\kappa)$, i.e. item (6) of Proposition 4.2.1, in the poset $\mathbb{P}_\delta$, for $\delta \geqslant 1$.

Remark 5.2.16. Recall the node projections from item (5) of Proposition 4.2.1. Note that each condition $f \in \mathsf{Col}(\omega, <\kappa)$ is finite, and thus each name for a node projection $\dot{\pi}^f(t)$ is now a name for a maximal node in a finite set. We may thus assume that it is simply a node in T, rather than a name. We denote this node by $\pi^f(t)$.

Recall that it follows from item (3) of the definition of residue system that the union of the collapse conditions occurring in a residue system is itself a collapse condition. Thus the following definition makes sense.

Definition 5.2.17. If $\vec{r}_E$ is a residue system for p into M^δ_α and $t \in T$ is a node, we denote by $\pi^E(t)$ the node projection $\pi^f(t)$ where

$$f = f^p_0 \cup \bigcup_{x \in E} f^{r_x}_0.$$

Compare the following lemma with the flexibility property for $\dot{T}$, i.e. item (6) of Proposition 4.2.1.

Lemma 5.2.18 (Flexibility Lemma). *Let $\delta < \kappa^+$, $p \in \mathbb{P}_\delta$ and $\alpha \in E^p_\delta$. Assume that $\vec{r}_E$ is a residue system for p into M^δ_α. Suppose that $t \in T$ is a node with $\ell(t) \leqslant \alpha$. For any node $\bar{t} \in \mathsf{wd}(t) \times \mathsf{ht}(t)$, then there is a condition $q \leqslant p$ and residue system $\vec{v}_E$ for q into M^δ_α with*

1. *the index set of $\vec{v}_E$ is equal to the index set of $\vec{r}_E$ and the root conditions are equal, $v_{(\delta,\alpha)} = r_{(\delta,\alpha)}$,*
2. *any extension of q that forces "$\pi^E(t) <_{\dot{T}} \bar{t}$" must also force "$\bar{t} <_{\dot{T}} t$".*

Proof. Look at the collapse condition $f^p_0 \in \mathsf{Col}(\omega, <\kappa)$. Let $g \leqslant f^p_0$ be as in item (6) from Proposition 4.2.1. Obtain q and $\vec{r'}_E$ from p and $\vec{r}_E$, respectively, by extending each condition $r_{(\gamma,\beta)}$ only at their collapse coordinate by letting

$$f_0^{v(\gamma,\beta)} := f_0^{r(\gamma,\beta)} \cup (g \cap M^\gamma_\beta),$$

for every $(\gamma, \beta) \in E$. It now follows from the choice of g that any extension of q that forces "$\pi^E(t) <_{\dot{T}} \bar{t}$" must also force "$\bar{t} <_{\dot{T}} t$". We need to show that $\vec{v}_E$ is a residue system for q into M^δ_α. We verify items (4) and (5). We show first that t cannot be in the range of any $f_\xi^{r(\gamma,\beta+)}$ and exit node from M^γ_β, for $(\gamma, \beta) \in E$. For if it was, the preimage of t would also be an exit node from M^γ_β. Since $\beta \in N_\xi^{r(\gamma,\beta+)}$, this would mean that the preimage of t should also be an element in $V_{\ell(t)}$, by definition of the poset, which is impossible since $\ell(t) \leqslant \alpha \leqslant \beta$. Thus, items (4) and (5) cannot fail by minimality of the extension $g \leqslant f^p_0$. □

The following lemma is the most substantial lemma of the section. It, together with a lemma that states the existence of residue systems (Lemma 5.2.20), will imply that each $\mathbb{P}_\delta$ is strongly proper with respect to each model in $\{M^\delta_\alpha : \alpha \in \mathcal{E}_\delta\}$. See Corollary 5.2.20.

Lemma 5.2.19. *Let $\delta < \kappa^+$. If $\vec{r}_E$ is a residue system for p into M^δ_α, then for every $(\gamma, \beta) \in E$, the condition $r_{(\gamma,\beta)}$ is a residue for $r_{(\gamma,\beta+)}$ into M^γ_β.*

Proof. The proof is by induction on δ. By the induction hypothesis and Lemma 5.2.13, it is enough to show that the root condition $r_{(\delta,\alpha)}$ is a residue for p, since $r_{(\delta,\alpha^+)} = p$. Let $w \in \mathbb{P}_\delta \cap M^\delta_\alpha$ extend $r_{(\delta,\alpha)}$. We find a common extension of w and p.

Base case and limit case:

The case $\delta = 1$ follows from the fact that the working parts of conditions in $\mathbb{P}_1$ are in $\mathsf{Col}(\omega, < \kappa)$, so the pointwise union $w \cup p$ is a common extension of w and p. The case when δ is a limit ordinal follows from the induction hypothesis: Let $\gamma < \delta$ be such that $\mathsf{sp}(w) \subseteq \gamma$. We can choose such γ in M^δ_α. By Lemma 4.1.5, $\gamma \cap M^\delta_\alpha \subseteq M^\gamma_\alpha$. Thus $w \restriction \gamma \in M^\gamma_\alpha$. Define the set E' from E by changing the pair (δ, α) to (γ, α), and define the system $\vec{r}_{E'}$ from $\vec{r}_E$ by replacing the root condition $r_{(\delta,\alpha)}$ by the condition $r_{(\gamma,\alpha)} := r_{(\delta,\alpha)} \restriction \gamma$. The resulting system is a residue system for $p \restriction \gamma$ into M^γ_α. Now $w \restriction \gamma$ extends the root condition $r_{(\gamma,\alpha)}$ so by the induction hypothesis, $w \restriction \gamma$ is compatible with $p \restriction \gamma$. Let $q \leqslant w \restriction \gamma, p \restriction \gamma$. Then $q^\frown p \restriction [\gamma, \delta)$ is a common extension of w and p.

Successor case $\boldsymbol{\delta + 1}$:

Now $w, p \in \mathbb{P}_{\delta+1}$. We will find a condition v in $\mathbb{P}_\delta$ that is a common extension of $w \restriction \delta$ and $p \restriction \delta$, and a function f extending the functions f^w_δ and f^p_δ, in such a way that $q := v^\frown(f, N^w_\delta \cup N^p_\delta)$ will be the desired common extension. For instance, this v must decide the meets in the set $\mathsf{dom}(f^w_\delta) \cup \mathsf{dom}(f^p_\delta)$, and the meets must be contained in the domain of f. The condition v and the function f will be defined recursively, by climbing up the models $M^\delta_{\beta_k}$. It is important to do this gradually, recursively, in order to avoid problems discussed in the introduction of this section. The main idea is the following: if we decide meets of nodes in $\mathsf{dom}(f^w_\delta)$ and a node $s \in \mathsf{dom}(f^p_\delta) - M^\delta_\alpha$ in a model M^δ_β that contains s but not its image $f^p_\delta(s)$, we do not touch the node projection $\pi(f^p_\delta(s))$, and are later free to modify the branch below $f^p_\delta(s)$ to correspond to the picture of the branch below s in the domain-side.

By minimality of E, we have $E_\delta = E^p_\delta - \alpha$. Enumerate E_δ in a strictly increasing order as

$$\alpha = \beta_0 < \cdots < \beta_{n-1}.$$

Denote $\beta_n := \kappa$, $r_n := p \restriction \delta$ and for each $k < n$, abbreviate

$$\begin{aligned} r_k &:= r_{(\delta,\beta_k)} \\ E_k &:= E \cap (M^\delta_{\beta_{k+1}} - M^\delta_{\beta_k}) \cap (\delta + 1 \times \kappa). \end{aligned}$$

We may assume that we added the pair (δ, α) to E_0 and denote $r_0 := r_{(\delta+1,\alpha)} \restriction \delta$. By Lemma 5.2.13, the system $\vec{r}_{E_k}$ is a residue system for r_{k+1} into $M^\delta_{\beta_k}$, and thus by the induction hypothesis, r_k is a residue of r_{k+1} into $M^\delta_{\beta_k}$. Also, for $k \leqslant n$, let

$$X_k := \{s \in \mathrm{dom}(f^p_\delta) : s \text{ is exit node from } M^\delta_\alpha \text{ and } f^p_\delta(s) \in M^\delta_{\beta_{k+1}} - M^\delta_{\beta_k}\}.$$

Then every node in $\mathrm{dom}(f^p_\delta)$ that is an exit node from M^δ_α belongs to some X_k, $k < n$, and $X_k \subseteq M^\delta_{\beta_k}$. The fact that $X_k \subseteq M^\delta_{\beta_k}$ follows from the fact that if $s \in X_k$, then $\ell(f^p_\delta(s)) \leqslant \beta_k$ and thus $s \in V_{\ell(t)} \subseteq M^\delta_{\beta_k}$.

We will define by recursion on $k \leqslant n$ conditions v_k and functions f_k.

Step 0: Let $v_0 := w \restriction \delta$ and $f_0 := f^w_\delta$.

Step $\boldsymbol{k+1}$: Assume that v_k and f_k are defined. Suppose that they satisfy:

1. $v_k \in \mathbb{P}_\delta \cap M^\delta_{\beta_k}$ and $v_k \leqslant r_k$.
2. f_k is a function in $M^\delta_{\beta_k}$ such that
$$v_k{}^\frown(f_k, N^w_\delta) \in \mathbb{P}_{\delta+1}$$
and the domain of f_k is the closure of the set $\mathrm{dom}(f^w_\delta) \cup \bigcup_{j<k} X_j$ under meets and under taking exit nodes from models $M^\delta_{\alpha'}$ where $\alpha' \in N^w_\delta$, as decided by v_k.

Up to extending v_k inside the model $M^\delta_{\beta_k}$, we may assume that it decides meets $s \wedge s'$ where $s \in X_k$ and $s' \in \mathrm{dom}(f^w_\delta)$, as well as their **implicit images**:

$$\boldsymbol{t_{s\wedge s'}} := \text{the node below } f^w_\delta(s') \text{ at the height of } s \wedge s',$$

as well as for every $\alpha' \in N^w_\delta$ and $s \in X_k$, the node

$$\boldsymbol{\bar{s}_{\alpha'}} := \text{the node } s' \leqslant s \text{ which is an exit node from } M^\delta_{\alpha'}.$$

Furthermore, up to extending v_k even further, by Lemma 5.2.4, we may assume that for each $\bar{s}_{\alpha'}$ such that $\bar{s}_{\alpha'} \neq s$, there is a node $t_{\bar{s}_{\alpha'}}$ such that if we let f'_k to be the function $f_k \cup \{(\bar{s}_{\alpha'}, \bar{t}_{\alpha'}) : \alpha' \in N^w_\delta, \bar{s}_{\alpha'} \neq s, s \in X_k\}$, then

$$v_k{}^\frown(f'_k, N^w_\delta) \in \mathbb{P}_{\delta+1}.$$

Since $\mathrm{wd}(f^p_\delta(s)) \in N^p_\delta$, we automatically have $t_{\bar{s}_{\alpha'}} \in \mathrm{wd}(f^p_\delta(s)) \times \mathrm{wd}(f^p_\delta(s))$ for every $s \in X_k$. Note also that $r_k \Vdash$ "$\pi^{E_k}(f^p_\delta(s)) \leqslant f_k(\tilde{s})$" if $\tilde{s} \notin \mathrm{dom}(f^{r(\delta+1,\alpha)}_\delta)$, for any $\tilde{s} \in \{s \wedge s' : s' \in \mathrm{dom}(f^w_\delta)\} \cup (\{\bar{s}_{\alpha'} : \alpha' \in N^w_\delta\} \cap V_\alpha)$.

The goal is to define v_{k+1} and f_{k+1}. We will work in the model $M^\delta_{\beta_{k+1}}$. We will first look at nodes in the set $f^p_\delta[X_k]$ and modify branches below them, as follows.

Recall that $\vec{r}_{E_k}$ is a residue system for r_{k+1} into $M^\delta_{\beta_k}$ with root condition r_k. By the Flexibility Lemma 5.2.18, since $\ell(t) \leqslant \beta_k$ for every $t \in f^p_\delta[X_k]$, we find a condition $r'_{k+1} \leqslant r_{k+1}$ and a residue system $\vec{r}'_{E_k} \leqslant \vec{r}_{E_k}$ such that:

1. the root conditions are equal $r'_{\delta,\beta_k} = r_k$,
2. for any $s \in X_k$, $s' \in \mathsf{dom}(f^w_\delta)$ and $\alpha' \in N^w_\delta$, any common extension of v_k and r'_{k+1} forces the following:
   (a) $f^p_\delta(s) \wedge f^w_\delta(s') = t_{s\wedge s'}$,
   (b) $t_{\bar{s}_{\alpha'}} \leqslant_{\dot{T}} f^p_\delta(s)$.

By the induction hypothesis r_k is a residue for r'_{k+1} and since $v_k \leqslant r_k$, it must be compatible with r'_{k+1}. Let $v_{k+1} \leqslant v_k, \tilde{r}$. Applying the induction hypothesis inside the model $M^\delta_{\beta_{k+1}}$, we may assume that $v_{k+1} \in \mathbb{P}_\delta \cap M^\delta_{\beta_{k+1}}$. Define then f_{k+1} to be the function consisting of the following sets:

- f_k,
- $\{(s \wedge s', t_{s\wedge s'}) : s \in X_k \text{ and } s' \in \mathsf{dom}(f^w_\delta)\}$,
- $\{(\bar{s}_{\alpha'}, t_{\bar{s}_{\alpha'}}) : s \in X_k, \alpha' \in N^w_\delta\}$.

Then f_{k+1} is an injective function and by construction v_{k+1} forces that it is level- and meet-preserving tree-embedding, and moreover,

$$v_{k+1}{}^\frown(f_{k+1}, N^w_\delta) \in \mathbb{P}_{\delta+1}.$$

This ends the recursion. Finally, look at the condition v_n and the function f_n. Define

$$q := v_n{}^\frown(f_n \cup f^p_\delta, N^w_\delta \cup N^p_\delta).$$

By construction, q is a condition in $\mathbb{P}_{\delta+1}$ that extends both w and p. Hence $r_{(\delta+1,\alpha)}$ is a residue of p. This ends the proof of Lemma 5.2.19.

□

Lemma 5.2.20. *If $\alpha \in E^p_\delta$, then p has a residue system into M^δ_α.*

Proof. The proof is by induction on $\delta < \kappa^+$. First of all, up to extending $p(0)$, we may assume that f_0^p is nice with respect to every $t \in \bigcup_{\gamma<\delta} \mathsf{ran}(f_\gamma^p)$, in the sense of item (6) of Proposition 4.2.1.

Base case $\delta = 1$: If $p \in \mathbb{P}_1$ and $\alpha \in E_\delta^p$, we may let the index set to be $E := \{(1, \alpha)\}$ and the condition $r_{(1,\alpha)}$ to be the pairwise intersection $(f_0^p \cap V_\alpha, N_0^p \cap \alpha)$. This system is as wanted.

Limit case δ: Suppose that δ is a limit ordinal. Let $p \in \mathbb{P}_\delta$ and $\alpha \in E_\delta^p$. Let $\gamma \in \delta \cap M_\alpha^\delta$ be such that $\mathsf{sp}(p) \cap M_\alpha^\delta \subseteq \gamma$. Such γ exists by Lemma 4.1.5. By the induction hypothesis $p \restriction \gamma$ has a residue system $\vec{r}_D$ into M_α^γ. We define E from D by replacing (γ, α) by (δ, α) and letting $r_{(\delta,\alpha)} := r_{(\gamma,\alpha)}{}^\frown(\varnothing, \varnothing), \ldots$. Then $r_{(\delta,\alpha)} \in \mathbb{P}_\delta \cap M_\alpha^\delta$ and the system $\vec{r}_E$ is as wanted.

Successor case $\delta + 1$:

Let $p \in \mathbb{P}_{\delta+1}$ and $\alpha \in E_{\delta+1}^p$. We build a residue system for p. First, enumerate the finite set $E_\delta^p - \alpha$ in a strictly increasing order as

$$\alpha = \beta_0 < \beta_1 < \cdots < \beta_{n-1}.$$

Abbreviate $\beta_n := \kappa$. For every $k < n$, denote

$$X_k := \{s \in \mathsf{dom}(f_\delta^p) : s \text{ is an exit node from } V_\alpha \text{ and } f_\delta^p(s) \in M_{\beta_{k+1}}^\delta - M_{\beta_k}^\delta\}.$$

Then $\ell(f_\delta^p(s)) \leqslant \beta_k$ for every $s \in X_k$ and thus $X_k \subseteq M_{\beta_k}^\delta$. We proceed by reverse recursion on $k \leqslant n$.

Let $r_n := p \restriction \delta$. At step $k < n$, suppose that r_{k+1} has been defined and satisfies $r_{k+1} \in \mathbb{P}_\delta \cap M_{\beta_{k+1}}^\delta$. By the induction hypothesis r_{k+1} has a residue system $\vec{r}_{E_k}$ into $M_{\beta_k}^\delta$. By working inside the model $M_{\beta_{k+1}}^\delta$, by elementarity, we may assume that the residue system $\vec{r}_{E_k}$ is an element of $M_{\beta_{k+1}}^\delta$. Denote its root condition $r_{(\delta,\beta_k)}$ by r_k. Then $r_k \in \mathbb{P}_\delta \cap M_{\beta_k}^\delta$. Up to extending r_k inside the model $M_{\beta_k}^\delta$, we may assume that it decides, for every $s \in X_k$, the "implicit preimage" of the node s, i.e. the predecessor $\bar{s}$ of s at the height of

$$t_{\bar{s}} := \pi^g(f_\delta^p(s)),$$

where $g = f_0^{r_{k+1}} \cup \bigcup_{x \in E_k} f_0^{r_x}$. Furthermore, up to extending the corodinate zero $f_0^{r_k}$, still inside the model $M_{\beta_k}^\delta$, we may assume that it is nice with respect to every $t \in X_k$ in the sense of item (6) of Proposition 4.2.1.

This ends the recursion. We have now chosen the conditions r_k, $k \leqslant n$, and residue system $\vec{r}_{E_k}$ for r_{k+1} into $M_{\beta_k}^\delta$, as well as nodes $(\bar{s}, \bar{t}_s)$ for each $s \in X_k$. We look at the final condition $r_0 \in \mathbb{P}_\delta \cap M^\delta_\alpha$ and define

$$f := (f^p_\delta \cap M^\delta_\alpha) \cup \bigcup_{k \leqslant n} \{(\bar{s}, \bar{t}_s) : s \in X_k\}.$$

Let

$$r := r_0{}^\frown(f, N^p_\delta \cap \alpha).$$

We first verify that r is a condition in $\mathbb{P}_{\delta+1} \cap M^{\delta+1}_\alpha$.

Claim 5.2.21. $r \in \mathbb{P}_{\delta+1} \cap M^{\delta+1}_\alpha$.

Proof of Claim 5.2.21. To see $r \in M^{\delta+1}_\alpha$, it suffices to see that r_0, f and $N^p_\delta \cap \alpha$ are in $M^{\delta+1}_\alpha$. This is clear for r_0 and $N^p_\delta \cap \alpha$. We argue that the function f is in $M^{\delta+1}_\alpha$. Since $\alpha \in N^p_\delta$, the model M^δ_α and thus also $M^{\delta+1}_\alpha$ must be closed under the function f^p_δ, whence $f^p_\delta \cap M^\delta_\alpha \in M^{\delta+1}_\alpha$. And since each $s \in X_k$, for $k \leqslant n$, is an exit node from M^δ_α, so are the images $f^p_\delta(s)$, and thus the pairs $(\bar{s}, \bar{t}_s)$ must belong to M^δ_α. Thus $f \in M^{\delta+1}_\alpha$. Hence $r \in M^{\delta+1}_\alpha$.

We then argue that $r \in \mathbb{P}_{\delta+1}$. Since $r \restriction \delta = r_0 \in \mathbb{P}_\delta$, it suffices to verify that it forces all the relevant items about f and $N^p_\delta \cap \alpha$ from the definition of the poset. This follows straightforwardly using the fact that r_0 is a residue for $p \restriction \delta$. For instance, to show that r_0 decides meets in the set $\mathsf{dom}(f)$, let $s, s' \in \mathsf{dom}(f)$. Note first that if $s, s' \in \mathsf{dom}(f^p_\delta)$, then there is automatically some $\bar{s} \in \mathsf{dom}(f)$ such that $p \restriction \delta \Vdash$ ``$s \wedge s' = \bar{s}$'' and r_0 must force the same thing, for otherwise it cannot be a residue of $p \restriction \delta$. On the other hand, if $s \notin \mathsf{dom}(f^p_\delta)$, then $s = \bar{s}_0$ for some $s_0 \in X_k$, and then

$$r_k \Vdash ``\bar{s}_0 \wedge s' = s_0 \wedge s'''.$$

This implies that there is $\bar{s} \in \mathsf{dom}(f^p_\delta) \cap M^\delta_\alpha$ such that

$$p \restriction \delta \Vdash ``\bar{s} = s_0 \wedge s'''.$$

And then r_0 must also force ``$\bar{s} = \bar{s}_0 \wedge s'''$, being a residue for both r_k and $p \restriction \delta$. Hence r_0 must decide meets in $\mathsf{dom}(f)$. The remaining properties are verified similarly.

Note that the last clauses in the definition of the poset regarding models M^δ_β, $\beta \in N^p_\delta \cap \alpha$ and exit nodes from them in $\mathsf{dom}(f)$ are vacuously satisfied due to the fact that we only added nodes below nodes in $\mathsf{dom}(f^p_\delta)$: for instance, if $s \in X_k$ and $\bar{s} \notin M^\delta_\beta$, for some $\beta \in N^p_\delta \cap \alpha$, then the unique exit node from M^δ_β below $\bar{s}$ is the same node as the unique exit node below s, which must be decided by $p \restriction \delta$ and thus by r_0 and must be in $\mathsf{dom}(f^p_\delta) \cap M^\delta_\alpha$, as p is a condition. This ends the proof of Claim 5.2.21.

□

We are ready to define the residue system for p into $M^\delta_{\alpha+1}$. The index set E will be the set $\bigcup_{k\leqslant n} E_k$ where the pair (δ,α) is replaced by the pair $(\delta+1,\alpha)$, and for each $(\gamma,\beta)\in E_k$ such that $\beta^+<\kappa$, the pair (γ,β_{k+1}) is added to E. For such pairs, we let $r_{(\gamma,\beta_{k+1})} := r_{k+1}\restriction\gamma$. Let $r_{(\delta+1,\alpha)} := r$. For $(\gamma,\beta)\in E_k$, we have already defined $r_{(\gamma,\beta)}$.

It can be seen that $\vec{r}_E$ is a residue system for p by the same argument as the proof of Lemma 5.2.13.

The other properties follow straightforwardly.

□

Remark 5.2.22. We may assume that if $\alpha\in E^p_\delta$, then there is a residue function $r:\mathbb{P}_\delta/p\to\mathbb{P}_\delta\cap M^\delta_\alpha$ that is order-preserving and satisfies that the residue $r(q)$ always extends the trace of q to M^δ_α.

Corollary 5.2.23. *For every $\delta<\kappa^+$ and $\alpha\in\mathcal{E}_\delta$, if $\alpha\in E^p_\delta$, then p has a residue into M^δ_α.*

Proof. By Lemma 5.2.20, p has a residue system $\vec{r}_E$ into M^δ_α. Its root condition $r_{(\delta,\alpha)}$ is a residue for p into M^δ_α, by Lemma 5.2.19. □

Corollary 5.2.24. *For every $\delta<\kappa^+$, the poset $\mathbb{P}_\delta$ is strongly proper with respect to each model in the set $\{M^\delta_\alpha:\alpha\in\mathcal{E}_\delta\}$.*

Proof. Let $p\in M^\delta_\alpha$. By Lemma 5.2.2 there is $q\leqslant p$ such that $\alpha\in E^q_\delta$. By Lemma 5.2.20, every extension $q'\leqslant q$ has a residue system $\vec{r}_E$ into M^δ_α. Its root condition $r_{(\delta,\alpha)}$ is a residue of q' into M^δ_α. □

Corollary 5.2.24 gives an unconditioned version of Node Density lemma:

Corollary 5.2.25 (Node Density). *Let $\delta<\kappa^+$. For every $p\in\mathbb{P}_{\kappa^+}$ and $s\in S_\delta$ there is $q\leqslant p$ such that $s\in\mathrm{dom}(f^q_\gamma)$.*

Proof. Follows from Corollary 5.2.24 by Lemma 5.2.4. □

Corollary 5.2.26. *The poset $\mathbb{P}_\delta$ is κ-strongly proper.*

Proof. Follows from Corollary 5.2.24 by Lemma 4.1.6. □

Corollary 5.2.27. *Let $G \subseteq \mathbb{P}_{\kappa^+}$ be a generic filter.*

1. $\aleph_1^{V[G]} = \kappa$,
2. $\aleph_2^{V[G]} = \kappa^+$,
3. $(2^\omega)^{V[G]} = (2^{\omega_1})^{V[G]} = \aleph_2^{V[G]}$.

Proof. The preservation of κ follows from κ-strong properness (Corollary 5.2.26). Since $\mathrm{Col}(\omega, < \kappa) \subseteq_c \mathbb{P}_{\kappa^+}$ we must have $\aleph_1^{V[G]} = \kappa$. The preservation of κ^+ follows from κ-cc: the poset $\mathbb{P}_{\kappa^+}$ is a direct limit of posets that have κ^+-cc (for the trivial reason that $|\mathbb{P}_\delta| = \kappa$ for every $\delta < \kappa^+$), and thus $\mathbb{P}_{\kappa^+}$ must also have κ^+-cc. Hence $\aleph_2^{V[G]} = \kappa^+$. The third item follows also from strong properness; each $\mathbb{P}_\delta$, $\delta < \kappa^+$, has complete subposets of size $< \kappa$, and thus the continuum is pushed up to $\aleph_2^{V[G]}$. ☐

Now, using Corollary 5.2.25, in $V^{\mathbb{P}_{\kappa^+}}$, the tree $\dot{T}$ is a wide κ-tree and for every $\gamma < \kappa^+$, there is an injective meet- and level-preserving tree-embedding

$$\dot{f}_\gamma : \dot{S}_\gamma \to \dot{T}.$$

We still need to make sure that $\dot{T}$ remains Aronszajn throughout the iteration.

5.3 Preserving Aronszajnness

The last part of the proof is to show that the tree $\dot{T}$ is still Aronszajn in the final extension by $\mathbb{P}_{\kappa^+}$. Before the proof, we need to prove a quotient version of strong properness in order to carry out a splitting argument. By Lemma 5.2.20 and its proof we assume that if $p \in \mathbb{P}_\delta$ and $\alpha \in E^p_\delta$, then there is an order-preserving residue map $\mathbb{P}_\delta / p \to \mathbb{P}_\delta \cap M^\delta_\alpha$.

5.3.1 Residue systems in quotients

Notation 5.3.1. For $\delta \leqslant \bar{\delta} < \kappa^+$ and $p \in \mathbb{P}_\delta$, let

$$E^p_{\bar{\delta}} := \{\alpha \in \mathcal{E}_{\bar{\delta}} : \alpha \in N^p_\gamma \text{ for every } \gamma \in \delta \cap M^{\bar{\delta}}_\alpha\}.$$

Lemma 5.3.2. *Let $\delta \leqslant \bar{\delta} < \kappa^+$ and $p \in \mathbb{P}_\delta$. If $\bar{\alpha} \in E^p_{\bar{\delta}}$, then p is strongly $(\mathbb{P}_\delta, M^{\bar{\delta}}_{\bar{\alpha}})$-generic.*

Proof. We show that if $\bar{\alpha} \in E^p_{\bar{\delta}}$, then p has a residue in $M^{\bar{\delta}}_{\bar{\alpha}}$. This implies the existence of a residue map $\mathbb{P}_\delta/p \to \mathbb{P}_\delta \cap M^{\bar{\delta}}_{\bar{\alpha}}$, since any extension $p' \leqslant p$ also must satisfy $\bar{\alpha} \in E^{p'}_{\bar{\delta}}$. To this end, note that $\bar{\alpha} \in E^p_{\bar{\delta}}$ implies the existence of a condition $q \in \mathbb{P}_{\bar{\delta}}$ such that $\bar{\alpha} \in E^q_{\bar{\delta}}$ and $q \restriction \delta = p$. Indeed, q can be defined by stretching p by for every $\xi \in [\delta, \bar{\delta})$, letting $q(\xi)$ to be either $(\varnothing, \{\bar{\alpha}\})$ if $\xi \in M^{\bar{\delta}}_{\bar{\alpha}}$, and $q(\xi) := (\varnothing, \varnothing)$ if $\xi \notin M^{\bar{\delta}}_{\bar{\alpha}}$. Then $q \in \mathbb{P}_{\bar{\delta}}$ and $\bar{\alpha} \in E^q_{\bar{\delta}}$ so q has a residue r into $M^{\bar{\delta}}_{\bar{\alpha}}$, and $r \restriction \delta$ is a residue for p into $M^{\bar{\delta}}_{\bar{\alpha}}$: if $w \in \mathbb{P}_\delta \cap M^{\bar{\delta}}_{\bar{\alpha}}$ extends $r \restriction \delta$, then $w^\frown r \restriction [\delta, \bar{\delta})$ extends r in $\mathbb{P}_{\bar{\delta}} \cap M^{\bar{\delta}}_{\bar{\alpha}}$ and is compatible with q and thus with p too, since r is a residue of q into $M^{\bar{\delta}}_{\bar{\alpha}}$. ☐

If G is a generic filter on a subposet of $\mathbb{P}$ and $p \in \mathbb{P}/G$, we say that a condition $r \in (\mathbb{P}/G) \cap M$ is a **residue of p into M in the quotient $\mathbb{P}/G$** if for every $w \in (\mathbb{P}/G) \cap M$ that extends r there is $q \in \mathbb{P}_\delta/G$ that extends w and p. If there is an order-preserving residue map $\mathbb{P}/p \to \mathbb{P} \cap M$ and $G \subseteq \mathbb{P} \cap M$ is a generic filter that contains a residue of p, then $p \in \mathbb{P}/G$, and if H is a $V[G]$-generic filter on $\mathbb{P}/G$, then in fact it is a V-generic filter on $\mathbb{P}$.

Lemma 5.3.3. *If there is a path from $M^{\bar{\delta}}_{\bar{\alpha}}$ to M^δ_α, then $\mathbb{P}_\delta \cap M^{\bar{\delta}}_{\bar{\alpha}} \subseteq \mathbb{P}_\delta \cap M^\delta_\alpha$.*

Proof. Follows from Lemma 4.1.9. ☐

Lemma 5.3.4. *Let $\delta \leqslant \bar{\delta} < \kappa^+$, $\bar{\alpha} \leqslant \alpha < \kappa$ and $p \in \mathbb{P}_\delta$. Suppose that $\alpha \in E^p_\delta$, $\bar{\alpha} \in E^p_{\bar{\delta}}$ and there is a path from $M^{\bar{\delta}}_{\bar{\alpha}}$ to M^δ_α. Let G be a generic filter on $\mathbb{P}_\delta \cap M^{\bar{\delta}}_{\bar{\alpha}}$ such that $p \in \mathbb{P}_\delta/G$. Assume that $\vec{r}_E$ is a residue system of p into M^δ_α whose root condition is in $\mathbb{P}_\delta/G$. Then for every $(\gamma, \beta) \in E$, the condition $r_{(\gamma,\beta)}$ is a residue for $r_{(\gamma,\beta^+)}$ into M^δ_α in the quotient $\mathbb{P}_\delta/G$.*

Proof. Note first that every member of the residue system must be in the quotient $\mathbb{P}_\delta/G$ because the root condition is in $\mathbb{P}/G$. Thus, arguing by induction, it suffices to show that the root condition $r_{(\delta,\alpha)}$ is a residue for p into M^δ_α in the quotient $\mathbb{P}_\delta/G$. Since G contains a residue of $r_{(\delta,\alpha)}$ into $M^{\bar{\delta}}_{\bar{\alpha}}$, we have $r_{(\delta,\alpha)} \in (\mathbb{P}_\delta \cap M^\delta_\alpha)/G$. Let $w \in (\mathbb{P}_\delta/G) \cap M^\delta_\alpha$ extend $r_{(\delta,\alpha)}$. Then since w is in $\mathbb{P}_\delta/G \cap M^\delta_\alpha = (\mathbb{P}_\delta \cap M^\delta_\alpha)/G$, there is a $V[G]$-generic filter H on $(\mathbb{P}_\delta \cap M^\delta_\alpha)/G$ that contains w and therefore $r_{(\delta,\alpha)}$ too. This H is a V-generic filter on $\mathbb{P}_\delta \cap M^\delta_\alpha$ because there is an order-preserving residue map $(\mathbb{P}_\delta \cap M^\delta_\alpha)/r_{(\delta,\alpha)} \to \mathbb{P}_\delta \cap M^{\bar{\delta}}_{\bar{\alpha}}$. Also, it extends G. And since H contains $r_{(\delta,\alpha)}$, which is a residue of p into M^δ_α, we have $p \in \mathbb{P}_\delta/H$. So there is a generic filter K on $\mathbb{P}_\delta/H$ that contains p. Now K is a V-generic filter on $\mathbb{P}_\delta$ that contains w and p. It follows that there is a common extension of w and p in H, and this common extension must be in $\mathbb{P}_\delta/G$ because H extends G. ☐

Lemma 5.3.5. *Let $\delta \leqslant \bar{\delta} < \kappa^+$, $\bar{\alpha} \leqslant \alpha < \kappa$ and $p \in \mathbb{P}_\delta$. Suppose that $\alpha \in E^p_\delta$, $\bar{\alpha} \in E^p_{\bar{\delta}}$ and there is a path from $M^{\bar{\delta}}_{\bar{\alpha}}$ to M^δ_α. Let G be a generic filter on $\mathbb{P}_\delta \cap M^{\bar{\delta}}_{\bar{\alpha}}$ such that $p \in \mathbb{P}_\delta/G$. Then p has a residue system into M^δ_α whose every condition is in $\mathbb{P}_\delta/G$.*

Proof. The proof is by induction on δ. We look at the successor case $\delta + 1$.

Let $H \subseteq \mathbb{P}_{\delta+1}/G$ be a $V[G]$-generic filter that contains p. Then H is a V-generic filter on $\mathbb{P}_\delta$ that extends G. For any $\gamma \leqslant \delta + 1$ and $\beta \leqslant \kappa$, denote

$$H^\gamma_\beta := H \cap \mathbb{P}_\gamma \cap M^\gamma_\beta.$$

Enumerate the set $E^p_\delta - \alpha$ as

$$\alpha = \beta_0 < \cdots < \beta_{n-1}.$$

Abbreviate $\beta_n := \kappa$ and $r_n := p \restriction \delta$. Now

$$H^\delta_\alpha = H^\delta_{\beta_0} \subseteq \cdots \subseteq H^\delta_{\beta_{n+1}}$$

and each $H^\delta_{\beta_k}$ is a V-generic filter on $\mathbb{P}_\delta \cap M^\delta_{\beta_k}$. We proceed by reverse recursion on $k < n$. Assume that we have defined r_{k+1} and $r_{k+1} \in \mathbb{P}_\delta/H^\delta_{\beta_k}$. By the induction hypothesis r_{k+1} has a residue system $\vec{r}_{E_k}$ into $M^\delta_{\beta_k}$ with every condition in $\mathbb{P}_\delta/H^\delta_{\beta_k}$. In particular, the root condition $r_k := r_{(\delta,\beta_k)}$ is an element in $H^\delta_{\beta_k}$. Let X_k be the collection of pairs $(s,t) \in f^p_\delta$ of exit nodes from V_α such that $t \in M^\delta_{\beta_{k+1}} - M^\delta_{\beta_k}$. For each pair $(s,t) \in X_k$, let $t_{\bar{s}} := \pi^{E_k}(t)$. Then $t_{\bar{s}}$ is a node in $T \cap V_\alpha$ since t is an exit node from V_α. The branch below s is decided by the poset $\mathbb{P}_\delta \cap M^\delta_{\beta_k}$, because $\mathbb{P}_\delta, \dot{S}_\delta, s \in M^\delta_{\beta_k}$. Therefore the branch below s is in $V[H^\delta_{\beta_k}]$. Let $\bar{s}$ be the predecessor of s at the height of the node $t_{\bar{s}}$, as decided by the generic $H^\delta_{\beta_k}$. Up to extending r_k inside the generic $H^\delta_{\beta_k}$, we may assume that it forces that $\bar{s}$ is the implicit preimage of $t_{\bar{s}}$, for each such pair (s,t). Then, since $r_k \in H^\delta_{\beta_k}$, in particular also $r_k \in \mathbb{P}_\delta/H^\delta_{\beta_{k-1}}$ and $r_k \in \mathbb{P}_\delta/G$.

Finally, suppose that we have defined r_1. Then $r_1 \in \mathbb{P}_\delta/H^\delta_{\beta_0}$. By the induction hypothesis r_1 has a residue system $\vec{r}_{E_1}$ into M^δ_α with root condition r_0 in $\mathbb{P}_\delta/H^\delta_\alpha$. Define

$$f := (f^p_\delta \cap V_\alpha) \cup \bigcup_{k \leqslant n} \{(\bar{s}, t_{\bar{s}}) : (s,t) \in X_k\},$$

let $N := N^p_\delta \cap \alpha$ and let

$$r := {r_0}^\frown(f, N).$$

We claim that $r \in H$. Note that $r_0 \in H$. Note also that f is a finite level- and meet-preserving function from $\dot{S}_\delta^{H_\alpha^\delta}$ to $\dot{T}^{H_\alpha^\delta}$. Thus, it suffices to show that $f \subseteq f_\delta^H$ and $N \subseteq N_\delta^G$, where

$$f_\delta^H = \bigcup_{q \in H} f_\delta^q,$$
$$N_\delta^H = \bigcup_{q \in H} N_\delta^q.$$

The fact that $N \subseteq N_\delta^H$ follows immediately from the fact that $p \in H$ and $N = N_\delta^p \cap \alpha$. We show that $f \subseteq f_\delta^H$. Note first that the function f_δ^H is an injective level- and meet-preserving tree-embedding from $\dot{S}_\delta^H$ into $\dot{T}_\delta^H$. Furthermore, since $p \in H$, it extends the finite partial function f_δ^p. Furthermore, if $(s, t) \in f_\delta^H$ and $\bar{s} <_{\dot{S}_\delta^H} s$ and $\bar{t} <_{\dot{T}^H} t$ are such that $\mathsf{ht}(\bar{s}) = \mathsf{ht}(\bar{t})$, then also $(\bar{s}, \bar{t}) \in f_\delta^H$. Now, if $(\bar{s}, \bar{t}) \in f$, then there are $(s, t) \in f_\delta^p \subseteq f_\delta^H$ such that $r_k \Vdash ``\bar{s} < s$ and $\bar{t} < t''$. Since $r_k \in H$ for each k, it follows that $f \subseteq f_\delta^H$.

We have thus shown that $r \in H$. Since $r \in M^{\delta+1}$, it follows that $r \in (\mathbb{P}_{\delta+1} \cap M_\alpha^{\delta+1})/G$. The system $\vec{r}_E$ where $E = \bigcup_{k \leqslant n} \cup \{(\delta+1, \alpha)\}$ and $r_{(\delta+1,\alpha)} := r$ is thus a residue system for p into $M_\alpha^{\delta+1}$ with every condition in the quotient $\mathbb{P}_{\delta+1}/G$. $\square$

5.3.2 No new branches

What follows in this section is heavily inspired by the argument in Laver-Shelah [16] that allows one to amalgamate partial specialization functions using "splitting". In the following definition, the case $s = s'$ is not excluded.

Definition 5.3.6. Let $\mathbb{P}$ be a poset and let $\dot{S}$ be a $\mathbb{P}$-name for a tree. A pair of conditions (p, q) **splits a pair of nodes (s, s')** if there are distinct nodes $\bar{s}$ and $\bar{s}'$ and an ordinal $\bar{\alpha}$ such that

1. $p \Vdash ``\bar{s} < s$ and $\mathsf{ht}(\bar{s}) = \bar{\alpha}''$,
2. $q \Vdash ``\bar{s}' < s'$ and $\mathsf{ht}(\bar{s}') = \bar{\alpha}''$.

In this case we say that p and q split the pair (s, s') with the pair $(\bar{s}, \bar{s}')$.

The weakly compact cardinal is used in the following lemma. We suppose without loss of generality that the name $\dot{S}$ in the statement of the lemma is a $\mathbb{P}_\delta$-name for a normal wide κ-Aronszajn tree on $\kappa \times \kappa$ such that $\Vdash_{\mathbb{P}_\delta}$ ``$\mathrm{Lev}_\beta(\dot{S}) = \kappa \times \{\beta\}$" for every $\beta < \kappa$.

Lemma 5.3.7. *Let $\delta < \kappa^+$ and $\alpha \in \mathcal{E}_\delta$. Suppose that $\dot{S} \in M^\delta_\alpha$ is a $\mathbb{P}_\delta$-name for a wide κ-Aronszajn tree and $G \subseteq \mathbb{P}_\delta \cap M^\delta_\alpha$ is a generic filter. Suppose that p and q are two conditions in the quotient $\mathbb{P}_\delta/G$. Then for any nodes $s, s' \in \dot{S}$ of limit height that are an exit nodes from V_α there are two extensions $\hat{p} \leqslant p$ and $\hat{q} \leqslant q$ in the quotient $\mathbb{P}_\delta/G$ that split the pair (s, s').*

Proof. Let $s, s' \in \dot{S}$ be nodes at a limit level that are exit nodes from V_α. We first claim that there are two conditions $p^L, p^R \leqslant p$ in $\mathbb{P}_\delta/G$ that split s. If not, then there is $b \in V[G]$ such that $p \Vdash$ ``b is the branch below s". Let $\bar{\alpha}$ be the height of s and let $\bar{\beta}$ be the width of s, i.e. the least $\beta \leqslant \kappa$ such that $b \subseteq \bar{\alpha} \times \beta$. We consider three cases.

- **Case 1:** $\bar{\alpha} = \alpha$. Now the branch b is a cofinal branch in the tree $(\dot{S} \cap V_\alpha)^G$. But since M^δ_α reflects all Π^1_1-statements with parameters in it, and indeed $\dot{S}, \mathbb{P}_\delta \in M^\delta_\alpha$, the tree $(\dot{S} \cap V_\alpha)^G$ must be a wide α-Aronszajn tree. Hence it is impossible to have b in $V[G]$.
- **Case 2:** $\bar{\alpha} < \alpha$ and $\bar{\beta} = \alpha$. Now the branch b induces a cofinal function from $\bar{\alpha}$ to $\bar{\beta}$ in $V[G]$, by the definition of $\bar{\beta}$. But this is not possible, because $\bar{\beta} = \alpha = \omega_2^{V[G]}$ and $\bar{\alpha} < \bar{\beta}$.
- **Case 3:** $\bar{\alpha} < \alpha$ and $\bar{\beta} < \alpha$. Let G^+ be a generic filter on $\mathbb{P}_\delta$ that extends G and contains p. Now b is a bounded subset of $M^\delta_\alpha[G^+]$ and so $b \in M^\delta_\alpha[G^+]$, by Lemma 4.1.7. Furthermore $H_\theta[G^+]$ satisfies that the branch b has a supremum in $\dot{S}^{G^+}$, namely the node s. Since $M^\delta_\alpha[G^+] \preccurlyeq H_\theta[G^+]$ (again by Lemma 4.1.7), the model $M^\delta_\alpha[G^+]$ must also contain a supremum of b. But this is impossible since the unique supremum of the branch b is the node s, and s is an exit node from M^δ_α, and thus also from $M^\delta_\alpha[G^+]$.

Hence we may find two extensions $p^L, p^R \leqslant p$ in the quotient $\mathbb{P}_\delta/G$ that split s at some height $\bar{\alpha} < \alpha$ with some distinct nodes s^L and s^R. In the quotient $\mathbb{P}_\delta/G$, find an extension $\hat{q} \leqslant q$ that decides the predecessor of s' at height $\bar{\alpha}$, call it $\bar{s}'$. If $s^L \neq \bar{s}'$, let $\hat{p} := p^L$, and otherwise let $\hat{p} := p^R$. Then $\hat{p}$ and $\hat{q}$ are as wanted.

□

Lemma 5.3.8. *Let $\delta < \kappa^+$, $\alpha \in \mathcal{E}_\delta$ and let $G \subseteq \mathbb{P}_\delta \cap M^\delta_\alpha$ be a generic filter. Assume that $r^L, r^R \in \mathbb{P}_\delta/G$ are two conditions and f and N satisfy*

1. *$r^{L^\frown}(f, N)$ and $r^{R^\frown}(f, N)$ are conditions in $\mathbb{P}_{\delta+1}$ and $\alpha \in N$,*
2. *r^L and r^R decide the widths of nodes in the set $\mathsf{dom}(f)$ similarly,*
3. *the conditions r^L and r^R split any pair of nodes from $\mathsf{dom}(f)$ that are exit nodes from V_α at a limit level with some nodes in $\mathsf{dom}(f) \cap V_\alpha$,*
4. *if $s \in \mathsf{dom}(f)$ is an exit node from V_α at a successor level, then r^L and r^R decide the immediate predecessor of s the same way and it is in $\mathsf{dom}(f) \cap V_\alpha$.*

Then the conditions $r^{L^\frown}(f, N)$ and $r^{R^\frown}(f, N)$ have a common residue into $M_\alpha^{\delta+1}$.

Proof. Note first that the function $f \cap V_\alpha$ is a level- and meet-preserving tree-embedding from $(\dot{S}_\delta \cap V_\alpha)^G$ to $(\dot{T} \cap V_\alpha)^G$, whose domain is closed under meets.

We build residue systems for $r^{L^\frown}(f, N)$ and $r^{R^\frown}(f, N)$ by reverse recursion on $k \leqslant n$. In the beginning, let $v_n^L := r^L$ and $v_n^R := r^R$. Then $v_n^L, v_n^R \in \mathbb{P}_\delta / G$.

At step $k < n$, suppose that v_{k+1}^L and v_{k+1}^R are defined and are in $(\mathbb{P}_\delta \cap M^\delta_{\beta_{k+1}})/G$. By Lemma 5.3.5, they have residue systems $\vec{v}^L_{F_k^L}$ and $\vec{v}^R_{F_k^R}$ into $M^\delta_{\beta_k}$ whose root conditions v_k^L and v_k^R are in the quotient $\mathbb{P}_\delta / G$. These residue systems can be chosen inside the model $M^\delta_{\beta_{k+1}}$. Exactly as in the proof of Lemma 5.2.20 we extend the root conditions v_k^L and v_k^R inside the model $M^\delta_{\beta_k}$ to decide, for every node s in the set

$$X_k := \{s \in \mathsf{dom}(f) : s \text{ is an exit node from } V_\alpha \text{ and } f(s) \in M^\delta_{\beta_{k+1}} - M^\delta_{\beta_k}\},$$

the implicit preimages $\bar{s}^L$ and $\bar{s}^R$ for the nodes $t_{\bar{s}^L} := \pi^{F_k^L}(f(s))$ and $t_{\bar{s}^R} := \pi^{F_k^R}(f(s))$, respectively. Note that for any two nodes $s, s' \in X_k$ (possibly $s = s'$), if $\bar{s}, \bar{s}' \in \mathsf{dom}(f)$ are distinct nodes such that

$$r^L \Vdash \bar{s}^L <_{\dot{S}_\delta} s \quad \text{and} \quad r^R \Vdash \bar{s}; <_{\dot{S}_\delta} s',$$

it then holds that

$$\bar{s} \leqslant_{(\dot{S}_\delta \cap V_\alpha)^G} s \quad \text{and} \quad \bar{s} \leqslant_{(\dot{S}_\delta \cap V_\alpha)^G} s'.$$

In particular the meet of $\bar{s}^L$ and $\bar{s}^R$ is the meet of $\bar{s}$ and $\bar{s}'$, and thus already an element in $\mathsf{dom}(f)$. Furthermore, the meet of $t_{\bar{s}^L}$ and $t_{\bar{s}'^R}$ must be the meet of the nodes $f(\bar{s})$ and $f(\bar{s}')$. So in particular the function

$$f_k := (f \cap V_\alpha) \cup \{(\bar{s}^L, t_{\bar{s}^L}), (\bar{s}^R, t_{\bar{s}^R}) : s \in \bigcup_{k \leqslant l < n} X_l\}$$

must be a level- and meet-preserving tree-embedding from $(\dot{S}_\delta \cap V_\alpha)^G$ to $(\dot{T} \cap V_\alpha)^G$ whose domain is closed under meets.

Finally, after the recursion, the function f_0 is a finite level- and meet-preserving tree-embedding from $(\dot{S} \cap V_\alpha)^G$ to $(\dot{T} \cap V_\alpha)^G$ that extends $f \cap V_\alpha$. Without loss of generality, $v_0 := v_0^L = v_0^R \in G$, and the concatenation $v_0{}^\frown(\hat{f}, N_\delta^p \cap \alpha)$ is a condition in $(\mathbb{P}_{\delta+1} \cap M_\alpha^{\delta+1})/G$. Let F^L be the union of the sets F_k^L, $k \leqslant n$, together with the pair $(\delta+1, \alpha)$ and similarly let F^R be the union of the F_k^R, $k \leqslant n$, together with the pair $(\delta+1, \alpha)$. Let

$$v^L_{(\delta+1,\alpha)} = v^R_{(\delta+1,\alpha)} = v_0{}^\frown(f_0, N \cap \alpha)$$

Then $\vec{v}^L_{F^L}$ and $\vec{v}^R_{F^R}$ are residue systems for $r^{L\frown}(f, N)$ and $r^{R\frown}(f, N)$, respectively, into $M_\alpha^{\delta+1}$, with the common root condition. The common root condition is a common residue for $r^{L\frown}(f, N)$ and $r^{R\frown}(f, N)$ into $M_\alpha^{\delta+1}$. □

Lemma 5.3.9. *Assume that the bookkeeping function is such that $\dot{S}_\gamma$ is a $\mathbb{P}_\gamma$-name for a wide κ-Aronszajn tree, for every $\gamma < \kappa^+$. Let $\delta < \kappa^+$.*

1. *For any $p \in \mathbb{P}_\delta$, $\alpha \in E_\delta^p$ and any finite set of nodes $A \subseteq \dot{T}$ of limit height that are exit nodes from M_α^δ, there are two conditions $q^L, q^R \leqslant p$ that have a common residue into M_α^δ and split every node in A.*
2. *$\dot{T}$ is a κ-Aronszajn tree in $V^{\mathbb{P}_\delta}$.*

Proof. The proof is by induction on δ. We prove item (1) first. The base case follows using Proposition 4.2.1(6). The limit case follows straightforwardly from the induction hypothesis. We consider the successor case $\delta + 1$.

Let $p \in \mathbb{P}_{\delta+1}$, $\alpha \in E^p_{\delta+1}$ and let $A \subseteq \dot{T}$ be a finite set of exit nodes from $M_\alpha^{\delta+1}$. Suppose up to extending p that whenever $s \in \mathrm{dom}(f_\delta^p)$ is an exit node from V_α at a successor height, then $p \restriction \delta$ decides its immediate predecessor and this predecessor is in $\mathrm{dom}(f_\delta^p)$. Let $\vec{r}_E$ be a residue system for p into $M_\alpha^{\delta+1}$. Enumerate the set E_δ as $\beta_0 < \cdots < \beta_{n-1}$, abbreviate $r_k := r_{(\delta,\beta_k)}$, denote $r_n := p \restriction \delta$, $\beta_n := \kappa$ and let

$$E_k := E \cap (M^\delta_{\beta_{k+1}} - M^\delta_{\beta_k}).$$

Then $\vec{r}_{E_k}$ is a residue system for r_{k+1} into $M^\delta_{\beta_k}$, for every $k < n$, by Lemma 5.2.13. Let $G \subseteq \mathbb{P}_\delta \cap M_\alpha^\delta$ be a generic filter that contains the condition r_0. Then $r_k \in (\mathbb{P}_\delta \cap M^\delta_{\beta_k})/G$ for each k and $r_{(\delta+1,\alpha)} \in (\mathbb{P}_{\delta+1} \cap M_\alpha^{\delta+1})/G$. For simplicity, we may suppose that

$f_\delta^{r(\delta+1,\alpha)}$ and $N_\delta^{r(\delta+1,\alpha)}$ are as in the proof of Lemma 5.2.20; $f_\delta^{r(\delta+1,\alpha)}$ is obtained from f_δ^p by adding pairs $(\bar{s}, t_{\bar{s}})$ obtained from pairs $(s,t) \in f_\delta^p$ of exit nodes from V_α by letting $t_{\bar{s}} := \pi^{E_k}(t)$, where k is such that $t \in M^\delta_{\beta_{k+1}} - M^\delta_{\beta_k}$, and $\bar{s}$ is forced by r_k to be the predecessor of s at the height of $t_{\bar{s}}$, and $N_\delta^{r(\delta+1,\alpha)} = N_\delta^p \cap \alpha$. It holds in particular that if r is any common extension of $r_0, \ldots, r_n$ and $p \restriction \delta$ in $\mathbb{P}_\delta$, then $r^\frown(f_\delta^{r(\delta+1,\alpha)} \cup f_\delta^p, N_\delta^p)$ is a condition in $\mathbb{P}_{\delta+1}$. For each $k < n$, let

$$X_k := \{s \in \mathsf{dom}(f_\delta^p) : s \text{ is an exit node from } V_\alpha \text{ and } f_\delta^p(s) \in (M^\delta_{\beta_{k+1}} - M^\delta_{\beta_k})\}.$$

Again, $X_k \subseteq M^\delta_{\beta_k}$ and if $s \in X_k$, then $\ell(f_\delta^p(s)) \leqslant \beta_k$. We build conditions $r_k^L, r_k^R \leqslant r_k$ in $(\mathbb{P}_\delta \cap M^\delta_{\beta_k})/G$ and functions f_k by recursion on $k \leqslant n+1$.

Let $r_0^L = r_0^R := r_0$ and $f_0 := f_\delta^{r(\delta+1,\alpha)}$. Suppose that r_k^L, r_k^R and f_k were defined. Suppose that they satisfy:

1. $r_k^L, r_k^R \in (\mathbb{P}_\delta \cap M^\delta_{\beta_k})/G$ and $r_k^L, r_k^R \leqslant r_k$,
2. r_k^L and r_k^R split every node in $A \cap M^\delta_{\beta_k}$,
3. r_k^L and r_k^R split every pair of nodes (s,s') from $\bigcup_{j \leqslant k} X_j$, with a pair nodes $(\bar{s}, \bar{s}')$,
4. f_k is a level- and meet-preserving tree-embedding from $(\dot{S}_\delta \cap V_\alpha)^G$ to $(\dot{T} \cap V_\alpha)^G$ such that extends $f_\delta^{r(\delta+1,\alpha)}$ and the domain satisfies

   $$\mathsf{dom}(f_\delta^{r(\delta+1,\alpha)}) \cup \{\bar{s}, \bar{s}' : (s,s') \in (\bigcup_{j \leqslant k} X_j)^2\}.$$

   and both r_k^L and r_k^R force that $f_k \cup f_\delta^p$ is a level- and meet-preserving tree-embedding from $\dot{S}_\delta$ to $\dot{T}$.

We build r_{k+1}^L, r_{k+1}^R and f_{k+1}.

Claim 5.3.10. *There are common extensions* $r_{k+1}^L \leqslant r_k^L, r_{k+1}$ *and* $r_{k+1}^R \leqslant r_k^R, r_{k+1}$ *in the quotient* $(\mathbb{P}_\delta \cap M^\delta_{\beta_{k+1}})/G$ *such that for every pair* $(s,s') \in \bigcup_{j \leqslant k} X_j$,

$$r_{k+1}^L \Vdash f_k(\bar{s}) <_{\dot{T}} f_\delta^p(s),$$
$$r_{k+1}^R \Vdash f_k(\bar{s}') <_{\dot{T}} f_\delta^p(s').$$

Proof. Using Lemma 5.2.18, as in the proof of Lemma 5.2.19. We get the common extensions in the quotient $(\mathbb{P}_\delta \cap M^\delta_{\beta_{k+1}})/G$ by Lemma 5.3.4. □

It now holds that both r^L_{k+1} and r^R_{k+1} force that the function

$$f_k \cup (f^p_\delta \cap M^\delta_{\beta_{k+1}})$$

is a level- and meet-preserving tree-embedding.

Claim 5.3.11. *There are extensions* $\tilde{r}^L_{k+1} \leqslant r^L_{k+1}$ *and* $\tilde{r}^R_{k+1} \leqslant r^R_{k+1}$ *in* $(\mathbb{P}_\delta \cap M^\delta_{\beta_{k+1}})/G$ *that split:*

1. *every node in* $A \cap M^\delta_{\beta_{k+1}}$*, and*
2. *every pair of nodes* (s, s') *from* $\bigcup_{j \leqslant k+1} X_j$ *with distinct nodes* $(\bar{s}, \bar{s}')$*.*

Proof. Using Lemma 5.3.7 repeatedly. □

Up to extending r^L_k and r^R_k in the quotient, we assume that they satisfy Claim 5.3.11. So any pair $(s, s') \in \bigcup_{j \leqslant k} X_j$ is split by a pair $(\bar{s}, \bar{s}')$ by r^L_k and r^R_k. Note that the meet $\bar{s} \wedge \bar{s}'$ is calculated in $V[G]$ in the tree $(\dot{S}_\delta \cap V_\alpha)^G$. It holds that:

- r^L_{k+1} forces $s \wedge \bar{s}' = \bar{s} \wedge \bar{s}'$, and
- r^R_{k+1} forces $\bar{s} \wedge s' = \bar{s} \wedge \bar{s}'$.

It follows that both r^L_{k+1} and r^R_{k+1} force that the set

$$(\mathsf{dom}(f^p_\delta) \cap V_{\beta_{k+1}}) \cup \{\bar{s}, \bar{s}' : (s, s') \in (\bigcup_{j \leqslant k} X_j)^2\}$$

is closed under meets. Consider the function f_k. By assumption it is a level- and meet-preserving tree-embedding from $(\dot{S}_\delta \cap V_\alpha)^G$ to $(\dot{T} \cap V_\alpha)^G$. By the Node Density Lemma 5.2.4 applied in $(\mathbb{P}_{\delta+1} \cap M^{\delta+1}_\alpha)/G$, there is a level- and meet-preserving tree-embedding $f_{k+1} \supseteq f_k$ whose domain is the set

$$\mathsf{dom}(f_k) \cup \{\bar{s}, \bar{s}' : (s, s') \in (\bigcup_{j \leqslant k} X_k)^2\},$$

Indeed, the domain of f_{k+1} must be closed under meets because the meet of $\bar{s}$ and $\bar{s}'$ in $(\dot{S}_\delta \cap V_\alpha)^G$ is the meet of s and s', as $p \upharpoonright \delta$, r^L_k and r^R_k all decide it, which must already be in the domain of the function $f^p_\delta \cap V_\alpha$. This ends the definition of r^L_{k+1}, r^R_{k+1} and f_{k+1}.

Finally, suppose that we have defined r_n^L and r_n^R. They are in $\mathbb{P}_\delta/G$ and extend each r_k, $k \leqslant n$. In particular, they extend $p \restriction \delta$. Define

$$f := f_\delta^p \cup f_n.$$

By construction, both r_n^L and r_n^R force that f is a level- and meet-preserving tree-embedding from $\dot{S}_\delta$ to $\dot{T}$. Let

$$\begin{aligned} q^L &:= r_n^{L\frown}(f, N_\delta^p), \\ q^R &:= r_n^{R\frown}(f, N_\delta^p). \end{aligned}$$

Furthermore, by construction, both q^L and q^R are conditions in $\mathbb{P}_{\delta+1}$. They split every node in A. They also split every pair (s, s') of exit nodes from V_α in f. By Lemma 5.3.8 q^L and q^R have residue systems $\vec{v}_{F^L}^L$ and $\vec{v}_{F^R}^R$ into $M_\alpha^{\delta+1}$ that have the same root condition $v_{(\delta+1,\alpha)}^L = v_{(\delta+1,\alpha)}^R$.

We prove item (2). Let $\delta < \kappa^+$ and suppose that the lemma holds for every $\gamma < \delta$. By what we just proved, item (1) holds for δ. Suppose to the contrary that there are $\dot{b} \in V^{\mathbb{P}_\delta}$ and $p \in \mathbb{P}_\delta$ such that

$$p \Vdash \text{``}\dot{b} \text{ is a cofinal branch in } \dot{T}\text{''}.$$

Let $\alpha \in \mathcal{E}_\delta$ be such that $p, \dot{b} \in M_\alpha^\delta$. Let $q \leqslant p$ be such that $\alpha \in E_\delta^q$. Up to extending q, assume that it decides the node $t := \dot{b}(\alpha)$. By item (1) there are two conditions $q^L, q^R \leqslant q$ that split t with some distinct nodes t^L and t^R at some height $\bar{\alpha} < \alpha$, and have a common residue into M_α^δ. Let r be the common residue. Find $w \in \mathbb{P}_{\delta+1} \cap M_\alpha^{\delta+1}$ that extends r and decides the node $\bar{t} := \dot{b}(\bar{\alpha})$. Then, if $\bar{t} \neq t^L$, w cannot be compatible with q^L, and if $\bar{t} \neq t^R$, w cannot be compatible with q^R. This contradicts the fact that r is a common residue for q^L and q^R, and ends the proof of item (2).

$\square$

We may conclude:

Theorem 5.3.12. *Assume that there exists a weakly compact cardinal. It is consistent that there is a universal wide* $\aleph_1$*-Aronszajn tree.*

Proof. Let $\kappa \to H(\kappa^+)$, $\gamma \mapsto \dot{S}_\gamma$ be a bookkeeping function such that for every $\mathbb{P}_{\kappa^+}$-name $\dot{S}$ for a wide κ-Aronszajn tree there is $\gamma < \kappa^+$ such that $\dot{S}_\gamma$ is a $\mathbb{P}_\gamma$-name for a wide κ-Aronszajn tree and $\Vdash_{\mathbb{P}_\gamma}$ "$\dot{S}_\gamma \cong \dot{S}$". Such a function exists by the fact that

$\mathbb{P}_{\kappa^+}$ has κ^+-cc. Then, in $V^{\mathbb{P}_{\kappa^+}}$, $\aleph_1$ is κ, and $\dot{T}$ is a wide $\aleph_1$-tree that is universal for all wide $\aleph_1$-Aronszajn trees. Also $\dot{T}$ must be Aronszajn, since otherwise by κ^+-cc there is $\delta < \kappa^+$ such that $\dot{T}$ is not Aronszajn in $V^{\mathbb{P}_\delta}$, which contradicts Lemma 5.3.9. $\square$

I conjecture that the forcing construction can be run over L using the combinatorial principle $\Diamond^\#$ from [2] and the filter derived from it, instead of collapsing a weakly compact cardinal and using the weakly compact filter, by replacing $\mathsf{Col}(\omega, < \kappa)$ by $\mathsf{Add}(\omega, \omega_1)$. This would mean that the existence of a universal wide $\aleph_1$-Aronszajn tree could be shown to be consistent without any large cardinal assumptions.

Chapter 6

Trees at double successor cardinals

In this chapter we show that the existence of a universal wide μ^+-Aronszajn tree is consistent for an uncountable successor cardinal μ, assuming the consistency of a weakly compact cardinal. The proof is the result of a collaboration with Professors Omer Ben-Neria, Menachem Magidor and Jouko Väänänen. The presentation here differs slightly from the preprint [1], but the changes are mostly cosmetic. Potential mistakes are mine. For simplicity, we present the proof in the case $\mu^+ = \aleph_2$, so the constructed tree will be a universal wide $\aleph_2$-Aronszajn tree.

6.1 The embedding poset

Assumption. *For the rest of the chapter, we assume that* GCH *holds in V. We also fix a weakly compact cardinal κ, a $\mathrm{Col}(\omega_1, <\kappa)$-name $\dot{T}$ for a wide κ-Aronszajn tree and width and labeling functions $t \mapsto \mathrm{wd}(t)$ and $t \mapsto \ell(t)$, as in Proposition 4.2.1, with $\mu = \omega_1$.*

As in Chapter 5, we fix a bookkeeping function that picks wide κ-trees, that are not necessarily yet assumed to be Aronszajn.

Notation 6.1.1. Fix a bookkeeping function $\kappa^+ \to H(\kappa^+)$, $\gamma \mapsto \dot{S}_\gamma$ and assume that whenever the poset $\mathbb{P}_\gamma$ is defined, then $\dot{S}_\gamma$ is a $\mathbb{P}_\gamma$-name for a normal tree with

$\Vdash_{\mathbb{P}_\gamma} \mathsf{Lev}_\alpha(\dot{S}_\gamma) = \kappa \times \{\alpha\}$. We denote by S_γ the domain of the tree $\dot{S}_\gamma$, i.e. the set $\kappa \times \kappa$.

We define the posets $\mathbb{P}_\delta$, $\delta \leqslant \kappa^+$, by recursion on δ. We follow the convention that $\mathbb{P}_0 = \{\varnothing\}$.

Notation 6.1.2. Assume that the posets $\mathbb{P}_\gamma, \gamma < \delta$, have already been defined for some $\delta \leqslant \kappa^+$. For each $\mathbb{P}_\gamma$, let $\mathcal{E}_\gamma$ and $(M^\gamma_\alpha : \alpha \in \mathcal{E}_\gamma)$ be as in Definition 4.1.4.

The models $(M^\gamma_\alpha : \alpha \in \mathcal{E}_\gamma)$, $\gamma < \delta$, are the ones that will be used in the definition of the poset $\mathbb{P}_\delta$. We assume without loss of generality that each model M^γ_α contains the bookkeeping function, as well as the label and width maps $t \mapsto \ell(t)$ and $t \mapsto \mathsf{wd}(t)$.

As in Chapter 5, if S is a tree and M is a set, then an **exit node** from M is a node $s \in S$ such that $s \notin M$ but the branch below s is contained in M. A subset $X \subseteq S$ is **closed under meets** if $s \wedge t \in X$ for any $s, t \in X$. For a set of ordinals Y, we denote by **lim** $\boldsymbol{Y}$ the set of limit points of Y that are in Y.

We are ready to define $\mathbb{P}_\delta$.

Definition 6.1.3. Let $\delta \leqslant \kappa^+$. Conditions in the poset $\mathbb{P}_\delta$ are functions

$$p : \delta \to V_\kappa$$

such that for every $\gamma < \delta$, $p(\gamma) = (f^p_\gamma, N^p_\gamma)$ satisfies

1. $f^p_0 \in \mathsf{Col}(\omega_1, <\kappa)$,
2. for non-zero $\gamma < \delta$, $f^p_\gamma : \kappa \times \kappa \to \kappa \times \kappa$ is a countable partial injective function such that
    (a) $p \restriction \gamma$ decides the $\dot{S}_\gamma$-meets in the set $\mathsf{dom}(f^p_\gamma)$,
    (b) $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ ``$\mathsf{dom}(f^p_\gamma)$ is closed under $\dot{S}_\gamma$-meets",
    (c) $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ ``$f^p_\gamma : \dot{S}_\gamma \to \dot{T}$ is a level- and meet-preserving tree-embedding",
3. $N^p_\gamma \subseteq \lim \mathcal{E}_\gamma$ is a countable set such that whenever $\alpha \in N^p_\gamma$ and $\xi \in \gamma \cap M^\gamma_\alpha$, then $\alpha \in N^p_\xi$, and such that the union
$$\bigcup_{\gamma<\delta} N^p_\gamma$$
is countable,

4. for every non-zero $\gamma < \delta$, every $s \in \mathsf{dom}(f^p_\gamma)$ and $\alpha \in N^p_\gamma$:
    (a) $s \in M^\gamma_\alpha$ if and only if $f^p_\gamma(s) \in M^\gamma_\alpha$,
    (b) $p \restriction \gamma \Vdash$ ``s is an exit node from M^γ_α if and only if $f^p_\gamma(s)$ is",
    (c) if $p \restriction \gamma \Vdash$ ``s is an exit node from M^γ_α", then $p \restriction \gamma$ decides the ordinal
    $$\mathsf{wd}(s) := \text{the least } \beta \in (\lim \mathcal{E}_\gamma - \mathsf{ht}(s)) \text{ such that } b_s \subseteq \beta \times \mathsf{ht}(s),$$
    and $f^p_\delta(s)$ satisfies $\mathsf{wd}(f^p_\delta(s)) = \mathsf{wd}(s)$,
    (d) if $s \notin M^\gamma_\alpha$, then there is $\bar{s} \in \mathsf{dom}(f^p_\gamma) - M^\gamma_\alpha$ such that
    $$p \restriction \gamma \Vdash \text{``}\bar{s} \leqslant_{\dot{S}_\gamma} s \text{ and } \bar{s} \text{ is an exit node from } M^\gamma_\alpha\text{"},$$
    (e) if $p \restriction \gamma \Vdash$ ``s is an exit node from M^γ_α", then the label $\ell(f^p_\gamma(s))$ satisfies:
        i. $\ell(f^p_\gamma(s)) \in \mathcal{E}_\gamma$,
        ii. $s \in M^\gamma_{\ell(f^p_\gamma(s))}$,
        iii. $\ell(f^p_\gamma(s)) \in N^p_\xi$ for every $\xi \in \gamma \cap M^\gamma_{\ell(f^p_\gamma(s))}$,
5. the support $\mathbf{sp}(\boldsymbol{p}) := \{\gamma \in \delta : f^p_\gamma \neq \varnothing\}$ is finite.

The order is the pointwise inverse inclusion: $q \leqslant p$ if $f^q_\gamma \supseteq f^p_\gamma$ and $N^q_\gamma \supseteq N^p_\gamma$ for all $\gamma < \delta$.

Remark 6.1.4. The only differences in the definition of the poset to build a universal wide $\aleph_2$-Aronszajn tree and the poset to build a universal wide $\aleph_1$-Aronszajn tree is that in the first case, the first poset is $\mathsf{Col}(\omega_1, < \kappa)$ and the conditions as well as the support are countable, whereas in the latter, the first poset is $\mathsf{Col}(\omega, < \kappa)$ and the conditions as well as the support are finite. To build a universal wide μ^+-Aronszajn tree for a successor cardinal μ, the first poset should be $\mathsf{Col}(\mu, < \kappa)$, and the conditions as well as the support should have size $< \mu$.

Remark 6.1.5. As in the case of the poset in Definition 5.1.4, also here it holds that if $\ell(t) \in \mathcal{E}_\delta$, then $t \notin M^\delta_{\ell(t)}$ and t is an exit node from $M^\delta_{\ell(t)}$. In particular, if $s \in \mathsf{dom}(f^p_\delta)$ is an exit node from M^δ_α for some $\alpha \in N^p_\delta$, and $f^p_\delta(s) = t$, then the model $M^\delta_{\ell(t)}$ "separates" the nodes s and t in the sense that

$$s \in M^\delta_{\ell(t)} \text{ and } t \notin M^\delta_{\ell(t)}.$$

In particular, it cannot happen that $\ell(t) \in N^p_\delta$, since each model with index in N^p_δ must be closed under the function f^p_δ. However, the model $M^\delta_{\ell(t)}$ must appear in the side conditions of the previous coordinates, i.e. it must hold that $\ell(t) \in N^p_\gamma$ for every $\gamma \in \delta \cap M^\delta_{\ell(t)}$.

Remark 6.1.6. Since each embedding approximation f^p_γ is forced to be meet-preserving and its domain is forced to be closed under meets, it follows that f^p_γ must be injective. See Lemma 2.1.2.

As in the case for the poset in Chapter 5, here too the set $\{\gamma < \delta : N^p_\gamma \neq \varnothing\}$ is not necessarily countable, even if the union $\bigcup_{\gamma<\delta} N^p_\gamma$ is. Also, it is important to note that whenever $p \in \mathbb{P}_\delta$ and $s \in \mathrm{dom}(f^p_\gamma)$ is an exit node from M^γ_α for some $\alpha \in N^p_\gamma$ and $t := f^p_\gamma(s)$, then the model $M^\gamma_{\ell(t)}$ separates s and t in the sense that

$$s \in M^\gamma_{\ell(t)} \text{ and } t \notin M^\gamma_{\ell(t)},$$

and the model $M^\gamma_{\ell(t)}$ occurs in the side conditions in the previous coordinates, i.e. if $\xi \in \gamma \cap M^\gamma_{\ell(t)}$, then $\ell(t) \in N^p_\xi$. But it necessarily holds that $\ell(t) \notin N^p_\gamma$, as $M^\gamma_{\ell(t)}$ is not closed under the function f^p_γ.

Notation 6.1.7. Each $\mathbb{P}_\delta$, $\delta < \kappa^+$, has size κ and is tacitly coded as a subset of V_κ, using the $<_\theta$-least bijection $\delta \to \kappa$.

Lemma 6.1.8. *If $\gamma < \delta$ $p \in \mathbb{P}_\delta$ and $q \in \mathbb{P}_\gamma$, and if $q \leqslant p \restriction \gamma$, then $q^\frown p \restriction [\gamma, \delta)$ is an extension of p. Thus $\mathbb{P}_\gamma \subseteq_c \mathbb{P}_\delta$.*

Lemma 6.1.9. *Let $\delta \leqslant \kappa^+$. If $p_n \in \mathbb{P}_\delta$, $n < \omega$, are such that $p_{n+1} \leqslant p_n$ for each $n < \omega$, then the pointwise union defined by*

$$p(\gamma) := \left(\bigcup_{n<\omega} f^{p_n}_\gamma, \bigcup_{n<\omega} N^{p_n}_\gamma\right)$$

for each $\gamma < \delta$, is a condition in $\mathbb{P}_\delta$ that extends each p_n, $n < \omega$.

Proof. Straightforward verification. □

We will see that the poset $\mathbb{P}_{\kappa^+}$ satisfies:

1. $\mathbb{P}_{\kappa^+}$ has κ^+-cc.
2. $\mathbb{P}_{\kappa^+}$ is countably closed, so preserves $\aleph_1$ and does not add reals.
3. $\mathbb{P}_{\kappa^+}$ collapses every $\alpha \in [\aleph_1, \kappa)$ onto $\aleph_1$.
4. $\mathbb{P}_{\kappa^+}$ preserves κ and so $V^{\mathbb{P}_{\kappa^+}} \models \aleph_2 = \kappa$.
5. $V^{\mathbb{P}_{\kappa^+}} \models 2^{\aleph_1} = \kappa^+ = \aleph_3$.

6. There is an injective level-preserving tree-embedding $\dot{f}_\gamma : \dot{S}_\gamma \to \dot{T}$ in $V^{\mathbb{P}_{\kappa^+}}$ for every $\gamma < \kappa^+$.
7. If the bookkeeping function picks only names for wide κ-trees that are Aronszajn, then $\dot{T}$ is a wide $\aleph_2$-Aronszajn tree.

The first property follows from the fact that $\mathbb{P}_{\kappa^+}$ is a direct limit of posets of size κ, i.e. posets that have κ^+-cc. The second property follows from Lemma 6.1.9. The third property follows from the fact that the first poset $\mathbb{P}_1$ contains $\mathsf{Col}(\omega_1, <\kappa)$ as a complete subposet. The rest of the section is devoted to proving the rest four properties.

6.2 Preservation of κ

We will show that each $\mathbb{P}_\delta$, $\delta < \kappa^+$, is strongly proper with respect to each model in the set $\{M^\delta_\alpha : \alpha \in \mathcal{E}_\delta\}$. This is done differently to the case of $\aleph_1$. The proof uses the fact that the posets are countably closed. We need the following notation:

Definition 6.2.1. Let $\delta < \kappa^+$ and $p \in \mathbb{P}_\delta$. Let

$$\boldsymbol{E^p_\delta} := \{\alpha \in \mathcal{E}_\delta : \alpha \text{ belongs to } N^p_\gamma \text{ for every } \gamma \in \delta \cap M^\delta_\alpha\}$$

We write E^p_γ for $E^{p\restriction\gamma}_\gamma$.

If $p \in \mathbb{P}_\delta$ and $\gamma < \delta$, then $N^p_\gamma \subseteq E^p_\gamma$. This follows from definitions. By assumption, N^p_γ consists of limit points of $\mathcal{E}_\gamma$, but E^p_γ might contain successor points of $\mathcal{E}_\gamma$ as well. So for $\gamma < \delta$, we always have $N^p_\gamma \subseteq E^p_\gamma$.

In order to show strong properness, we will show the following:

- If $p \in \mathbb{P}_\delta \cap M^\delta_\alpha$, then there is $q \leqslant p$ with $\alpha \in E^q_\delta$.
- If $\alpha \in E^p_\delta$, then there is $q \leqslant p$ that is *super-nice with respect to* M^δ_α.
- If p is super-nice with respect to M^δ_α, then its *trace* to M^δ_α is its residue into M^δ_α.

The first item is easy and the two last items require a bit more work. We will begin with the first item and then move to define traces and the class of conditions that are super-nice with respect to a model.

Lemma 6.2.2. *Let $\delta < \kappa^+$ and $\alpha \in \mathcal{E}_\delta$. If $p \in \mathbb{P}_\delta \cap M^\delta_\alpha$, then there is $q \leqslant p$ with $\alpha \in E^q_\delta$.*

Proof. Let $q : \delta \to V_\kappa$ be defined by

$$q(\gamma) := \begin{cases} (f^p_\gamma, N^p_\gamma \cup \{\alpha\}) & \text{if } \gamma \in \delta \cap M^\delta_\alpha, \\ p(\gamma) & \text{if } \gamma \notin \delta \cap M^\delta_\alpha. \end{cases}$$

Then q is a condition in $\mathbb{P}_\delta$ extending p such that $\alpha \in E^q_\delta$. This follows using the fact that $p \in M^\delta_\alpha$, which implies that $\mathrm{sp}(p) \subseteq M^\delta_\alpha$ and $f^p_\gamma \subseteq M^\delta_\alpha$ for every $\gamma < \delta$. ☐

We move to define traces and super-nice conditions and prove that the trace of a super-nice condition is a condition, as well as the second item of the above list. Before this, we collect an easy node density lemma that allows to add nodes to the embedding approximation part of a condition, given that they are forced to be below nodes already in the embedding approximation.

Lemma 6.2.3. *Let $\delta \leqslant \kappa^+$, $p \in \mathbb{P}_\delta$ and $\gamma < \delta$. Assume that $(s,t) \in f^p_\gamma$ and $(\bar{s},\bar{t})$ is a pair of nodes that satisfies:*

- $\bar{s} \in S_\gamma$, $\bar{t} \in T$ *and* $\mathrm{ht}(\bar{s}) = \mathrm{ht}(\bar{t})$,
- $p \restriction \gamma \Vdash$ "$\bar{s} <_{\dot{S}_\gamma} s$ *and* $\bar{t} <_{\dot{T}} t$".

Then q obtained from p by letting

$$f^q_\gamma := f^p_\gamma \cup \{(\bar{s},\bar{t})\},$$

$N^q_\gamma := N^p_\gamma$ and $q(\xi) := p(\xi)$ for $\xi \in \delta - \{\gamma\}$, is a condition in $\mathbb{P}_\delta$ that satisfies $q \leqslant p$.

Proof. Straightforward verification of the items in Definition 6.1.3. ☐

6.2.1 Traces and super-nice conditions

Definition 6.2.4. Let $\delta < \kappa^+$ and let M be a set. The **trace** $[p]_M$ of a condition $p \in \mathbb{P}_\delta$ to M is given by

$$[p]_M(\gamma) := \begin{cases} (f^p_\gamma \restriction M, N^p_\gamma \cap M) & \text{if } \gamma \in \delta \cap M, \\ (\varnothing, \varnothing) & \text{if } \gamma \notin \delta \cap M. \end{cases}$$

In general, the trace of a condition is not necessarily a condition. However, we will show that when p is *super-nice with respect to M^δ_α* (defined below, see Definition 6.2.9) for some $\alpha \in \mathcal{E}_\delta$, then the trace $[p]_{M_\alpha^\delta}$ is a condition in $\mathbb{P}_\delta \cap M_\alpha^\delta$ and moreover it is a residue for p into M_α^δ. See Corollary 6.2.19. Abusing the notation, we still write $q \leqslant [p]_\alpha^\delta$ whenever $f_\gamma^q \supseteq f_\gamma^p \cap M_\alpha^\delta$ and $N_\gamma^q \supseteq N_\gamma^p \cap M_\alpha^\delta$ for every $\gamma \in \delta \cap M_\alpha^\delta$.

Lemma 6.2.5. *Let $\delta < \kappa^+$ and let $\alpha, \beta \in \mathcal{E}_\delta$ be such that $\alpha < \beta$. For any $p \in \mathbb{P}_\delta$:*

1. $[[p]_\beta^\delta]_\alpha^\delta = [p]_\alpha^\delta$,
2. *if $\gamma \in \delta \cap M_\alpha^\delta$, then $[p \restriction \gamma]_\alpha^\gamma = [p]_\alpha^\delta \restriction \gamma$,*
3. $[p]_\alpha^\delta \in M_\alpha^\delta$.

Proof. The first item follows from the definition and the second item follows from Lemma 4.1.5 according to which $\gamma \cap M_\alpha^\delta = \gamma \cap M_\alpha^\gamma$. The third item follows from the fact that $[p]_\alpha^\delta$ is a countable subset of M_α^δ and therefore a member of it, as M_α^δ is closed under countable sequences. ☐

Recall the projections of nodes from Proposition 4.2.1: Given a node t which is an exit node from some model M_α^δ, each condition $f \in \mathsf{Col}(\omega_1, < \kappa)$ decides only countably many elements that are below t. The tree $\dot{T}$ is $< \mu$-closed, so these have a supremum, and $\dot{\pi}^f(t)$ is a name for this supremum. We write

$$\dot{\pi}^p(t) := \dot{\pi}^{f_0^p}(t),$$

for a condition $p \in \mathbb{P}_\delta$.

Recall the notions of a path and close from Definitions 4.1.8 and 4.1.10. We relativise paths and closures to conditions:

Definition 6.2.6. Let $\delta < \kappa^+$ and $p \in \mathbb{P}_\delta$.

1. A path $((\delta_k, \alpha_k) : k \leqslant n)$ is a **path in p** if $\alpha_k \in E_{\delta_k}^p$ for every $k \leqslant n$.
2. The **p-closure of M_α^δ** is the set of pairs (γ, β) such that there is a path in p from M_α^δ to M_β^γ.
3. The **p-closure** of a set of models $\{M_{\alpha_i}^{\delta_i} : i \in I\}$ with $\alpha_i \in E_\delta^p$ is the union of p-closures of $M_{\alpha_i}^{\delta_i}$ for $i \in I$.

Notation 6.2.7. The p-closure of a set of models is tacitly ordered lexicographically (starting in the second coordinate), i.e. $(\gamma, \beta) < (\gamma', \beta')$ if $\beta < \beta'$ or if $\beta = \beta'$ and $\gamma < \gamma'$.

Recall δ-sequences from Definition 4.1.11. We have an analogue of Lemma 4.1.12:

Lemma 6.2.8. *Let $\delta < \kappa^+$ and $p \in \mathbb{P}_\delta$. The p-closure of a δ-sequence is again a δ-sequence.*

Proof. Clear from definitions. ☐

Definition 6.2.9. Let $\delta < \kappa^+$ and $\alpha \in \mathcal{E}_\delta$.

1. A condition $p \in \mathbb{P}_\delta$ is **super-nice with respect to M^δ_α** if $\alpha \in E^p_\delta$ and whenever there is a path in p from M^δ_α to a model M^γ_β, then for every non-zero $\xi \in \gamma \cap M^\gamma_\beta$ and $s \in \mathrm{dom}(f^p_\xi)$ such that
$$p \restriction \xi \Vdash ``s \text{ is an exit node from } \beta \times \beta",$$
there are nodes $s_n \in \mathrm{dom}(f^p_\xi)$, $n < \omega$, such that
$$p \restriction \xi \Vdash `` \sup_n f^p_\xi(s_n) = \dot\pi^p(f^p_\xi(s))",$$
and moreover, if s is at a successor height, then its immediate predecessor belongs to the domain of f^p_ξ.
2. A condition $p \in \mathbb{P}_\delta$ is **fully super-nice** if it is super-nice with respect to M^δ_α for every $\alpha \in E^p_\delta$.

Lemma 6.2.10.

1. *If p is super-nice with respect to M^δ_α and $\gamma \in \delta \cap M^\delta_\alpha$, then $p \restriction \gamma$ is super-nice with respect to M^γ_α.*
2. *If p is super-nice with respect to M^δ_α and (γ, β) belongs in the p-closure of M^δ_α, then $p \restriction \gamma$ is super-nice with respect to M^γ_β.*

Proof. Follow from the definitions of super-nice conditions and p-closure of M^δ_α. ☐

The next goal is to show that the set of conditions that are super-nice with respect to M^δ_α are dense below any condition p with $\alpha \in E^p_\delta$. The proof is a dove-tailing argument. We prove it in two lemmas.

Lemma 6.2.11. *Let $\delta < \kappa^+$, $p \in \mathbb{P}_\delta$. Let $\gamma \in \mathrm{sp}(p)$, $\beta \in E^p_\gamma$ and $(s, t) \in f^p_\gamma$. Suppose that (s, t) are forced by $p \restriction \gamma$ to be exit nodes from M^γ_β. There is $q \leqslant p$ such that either*

1. *there is* $\bar{s} \in \mathrm{dom}(f^q_\gamma)$ *such that* $q \restriction \gamma \Vdash$ "$f^q_\gamma(\bar{s}) = \dot{\pi}^p(t)$", *or*
2. *there are* $\{(\bar{s}_n, \bar{t}_n) : n < \omega\} \subseteq f^q_\gamma$ *such that* $q \restriction \gamma \Vdash$ "$\sup_n \bar{t}_n = \dot{\pi}^p(t)$".

Proof. We proceed by recursion on $n < \omega$. Let $p_0 := p$. Whenever p_n is defined, fix a countable set of nodes t^n_m, $m < \omega$, such that

$$p_n \Vdash \sup_{m<\omega} t^n_m = \dot{\pi}^{p_n}(t),$$

and let $p_{n+1} \leqslant p_n$ be such that

1. p_{n+1} decides the **implicit preimage** for each t^n_m: i.e. node s^n_m such that

   $$p_{n+1} \Vdash \text{``}s^n_m \text{ is the unique predecessor of } s \text{ at the height of } t^n_m\text{''},$$

   for each $m < \omega$,
2. $(s^n_m, t^n_m) \in f^{p_{n+1}}_\gamma$ for every $m < \omega$.

Such a condition p_{n+1} can be found by first extending $p_n \restriction \gamma$ countably many times to to some $p' \leqslant p_n \restriction \gamma$ in $\mathbb{P}_\gamma$ that decides the nodes s^n_m, $m < \omega$, and then applying Lemma 7.2.14 to the concatenation $p'^\frown p_n \restriction [\gamma, \delta)$.

Finally, let q be the pointwise union of the p_n's. It is a condition, it extends p and it satisfies the lemma. ☐

Lemma 6.2.12 (Super-nice density). *Let* $\delta < \kappa^+$.

1. *If* $p \in \mathbb{P}_\delta$ *and* $\alpha \in E^p_\delta$, *then the set of conditions that are super-nice with respect to* M^δ_α *is dense below* p *and countably closed.*
2. *The set of conditions* $p \in \mathbb{P}_\delta$ *that are fully super-nice is dense and countably closed.*

Proof. It is clear that if $(p_n : n < \omega)$ is a descending sequence of conditions each of which is super-nice with respect to a model M^δ_α, then their pointwise union defined by

$$p(\gamma) = \left(\bigcup_{n<\omega} f^{p_n}_\gamma, \bigcup_{n<\omega} N^{p_n}_\gamma \right)$$

is a condition that extends each p_n and is super-nice with respect to M^δ_α. This shows that the set of conditions that are super-nice (with respect to a model or fully) is countably closed. We show the second item since it implies the first item.

Let $p \in \mathbb{P}_\delta$. Using a suitable pairing function $\langle\cdot,\cdot\rangle : \omega \times \omega \to \omega$ and Lemma 6.2.11, we find a countable sequence of conditions $(p_n : n < \omega)$ whose pointwise union will be fully super-nice. Namely, let $p_0 := p$ and suppose that p_n was defined. First fix an enumeration

$$((\xi^n_m, (\gamma^n_m, \beta^n_m), (s^n_m, t^n_m)) : m < \omega)$$

of all the tuples that satisfy:

- (γ^n_m, β^n_m) belongs to the p-closure of any model M^δ_α where $\alpha \in E^p_\delta$,
- $\gamma^n_m \in \gamma^n_m \cap M^{\gamma^n_m}_{\beta^n_m}$,
- $(s^n_m, t^n_m) \in f^{p_n}_{\xi^n_m}$ are forced by $p_n \restriction \xi^n_m$ to be exit nodes from $M^{\gamma^n_m}_{\beta^n_m}$.

Then look at the unique m, k such that $\langle m, k\rangle = n$. By changing $\langle\cdot,\cdot\rangle$ if necessary, we may assume that $k \leqslant n$. By Lemma 6.2.11, there is $p_{n+1} \leqslant p_n$ such that $\mathsf{ran}(f^p_{\xi^k_m})$ has nodes that are cofinal in $\dot{\pi}^{p_n}(t^k_m)$.

Let q be the pointwise union of the conditions p_n, $n < \omega$. Then q is a condition in $\mathbb{P}_\delta$ that extends p and is fully super-nice. This ends the lemma.

□

Lemma 6.2.13. *Let $\delta < \kappa^+$. Assume that it holds for every $\gamma < \delta$ and $\beta \in \mathcal{E}_\gamma$ that if $p \in \mathbb{P}_\gamma$ is super-nice with respect to M^γ_β, then its trace $[p]_{M^\gamma_\beta}$ is a condition in $\mathbb{P}_\gamma \cap M^\gamma_\beta$ and a residue of p into M^γ_β. Then it holds for every $\alpha \in \mathcal{E}_\delta$ that if $p \in \mathbb{P}_\delta$ is a condition that is super-nice with respect to M^δ_α, then its trace $[p]_{M^\delta_\alpha}$ is a condition in $\mathbb{P}_\delta \cap M^\delta_\alpha$.*

Proof. It is clear that that the trace $[p]_{M^\delta_\alpha}$ is (definable from[1]) a countable subset of M^δ_α, and therefore $[p]_{M^\delta_\alpha} \in M$. We show that $[p]_{M^\delta_\alpha} \in \mathbb{P}_\delta$. Firstly, since $\alpha \in E^p_\delta$, it follows that $\alpha \in N^p_\gamma$ for every $\gamma \in \delta \cap M^\delta_\alpha$, which means that the model M^δ_α is closed for the function f^p_γ for every $\gamma \in \delta \cap M^\delta_\alpha$. Thus $f^{[p]_{M^\delta_\alpha}}_\gamma$ is an injective function in M^δ_α. We need to show that $[p]_{M^\delta_\alpha} \restriction \gamma$ forces the relevant things about it, for instance that $[p]_{M^\delta_\alpha} \restriction \gamma$ decides meets in $\mathsf{dom}(f^{[p]_{M^\delta_\alpha}}_\gamma) = \mathsf{dom}(f^p_\gamma) \cap M^\delta_\alpha$.

---

[1] Strictly speaking $[p]_{M^\delta_\alpha}$ is a function with domain δ and in general $\delta \not\subseteq M^\delta_\alpha$, but since $p(\gamma) = (\varnothing, \varnothing)$ whenever $\gamma \notin M^\delta_\alpha$, this is irrelevant.

The assumption of the lemma combined with Lemma 6.2.5 ensures that $[p]_{M^\delta_\alpha} \restriction \gamma$ is a residue of $p \restriction \gamma$ for each $\gamma < \delta$ into M^δ_α. Hence, the following holds for any $\gamma < \delta$:

> *If φ is a statement in the forcing language of $\mathbb{P}_\gamma$ with parameters in M^δ_α and $p \restriction \gamma \Vdash \varphi$, then $[p]_{M^\delta_\alpha} \restriction \gamma \Vdash \varphi$.* $(*)$

Now $(*)$ holds because otherwise in M^δ_α, there is an extension $w \leqslant [p]_{M^\delta_\alpha} \restriction \gamma$ that forces $\neg\varphi$. This w cannot be compatible with $p \restriction \gamma$, which contradicts the fact that $[p]_{M^\delta_\alpha} \restriction \gamma$ is a residue of $p \restriction \gamma$.

It now follows easily from $(*)$ that $[p]_{M^\delta_\alpha} \restriction \gamma$ decides meets in the set $\mathsf{dom}(f_\gamma^{[p]_{M^\delta_\alpha}})$. The other requirements follow easily from $(*)$ as well. We conclude that $[p]_{M^\delta_\alpha}$ indeed is a condition in $\mathbb{P}_\delta \cap M^\delta_\alpha$.

□

6.2.2 Strong properness

The goal of this section is to show that if $p \in \mathbb{P}_\delta$ is super-nice with respect to M^δ_α (see Definition 6.2.9), then $[p]_{M^\delta_\alpha}$ is a residue of p into M^δ_α. This means finding a common extension of p and any $w \in \mathbb{P}_\delta \cap M^\delta_\alpha$ that extends the trace $[p]^\delta_\alpha$. This common extension will be built recursively, by climbing up the p-closure of M^δ_α. In order to carry out the proof, we need a strong induction hypothesis which is formulated in terms of "multi-extensions" (Definition 6.2.14).

In what follows, we will tacitly identify $\mathbb{P}_\gamma$ as a subposet of $\mathbb{P}_\delta$ whenever $\gamma < \delta$, and write $q \leqslant p$ for $q \restriction \gamma \leqslant p$ for $q \in \mathbb{P}_\delta$ and $p \in \mathbb{P}_\gamma$.

Definition 6.2.14. Let $\delta < \kappa^+$ and $p \in \mathbb{P}_\delta$. A sequence $(w_j : j < \iota)$ is a ***p*-multi-extension with respect to a $\boldsymbol{\delta}$-sequence $((\boldsymbol{\delta_j, \alpha_j}) : \boldsymbol{j < \iota})$** if

1. $p \restriction \delta_j$ is super-nice with respect to $M^{\delta_j}_{\alpha_j}$ for every $j < \iota$.
2. The sequence $((\delta_j, \alpha_j) : j < \iota)$ is a δ-sequence of countable length and an initial segment of its p-closure.
3. $w_j \in \mathbb{P}_{\delta_j} \cap M^{\delta_j}_{\alpha_j}$ and $w_j \leqslant [p \restriction \delta_j]^{\delta_j}_{\alpha_j}$.
4. If $i < j < \iota$, then $w_j \restriction \min\{\delta_i, \delta_j\} \leqslant w_i \restriction \min\{\delta_i, \delta_j\}$.

5. In case $\delta_j < \delta$: for any $s \in \mathsf{dom}(f^p_{\delta_j}) \cap V_{\alpha_j}$, any

$$s' \in \bigcup\{\mathsf{dom}(f^{w_i}_{\delta_j}) : i < j \text{ and } \delta_j < \delta_i\},$$

   and any $\beta \in \bigcup\{N^{w_i}_{\delta_j} : i < j \text{ and } \delta_j < \delta_i\}$,

   (a) w_j decides the $\dot{S}_{\delta_j}$-meet $s \wedge s'$ and its **implicit image**, i.e. the node

$$t_{s \wedge s'} := \text{the node below } f^{w_i}_{\delta_j}(s') \text{ at the height of } s \wedge s',$$

   (b) w_j decides the exit node

$$\bar{s}_\beta := \text{the unique node below } s \text{ that is an exit node from } M^{\delta_j}_\beta.$$

6. if $j' < j$ and $(s, t) \in f^p_{\delta_{j'}} \cap V_{\alpha_j}$, then for any

$$(s', t') \in \bigcup\{f^{w_i}_{\delta_{j'}} : i < j \text{ and } \delta_{j'} < \delta_i\},$$

   and any $\beta \in \bigcup\{N^{w_i}_{\delta_{j'}} : i < j \text{ and } \delta_{j'} < \delta_i\}$,

   (a) the collapse part $f^{w_j}_0$ forces

$$t_{s \wedge s'} = t \wedge t',$$

   where $t_{s \wedge s'}$ is the implicit image from the previous item (5),

   (b) there is a node $t_{\bar{s}_\beta} \in T$ at the height of $\bar{s}_\beta$ such that w_j forces

      i. $t_{\bar{s}_\beta} <_{\dot{T}} t$,
      ii. $t_{\bar{s}_\beta}$ is an exit node from $M^{\delta_{j'}}_\beta$
      iii. $\mathsf{wd}(t_{\bar{s}_\beta}) = \mathsf{wd}(\bar{s}_\beta)$.

We will show that if $(w_j : j < \iota)$ is a p-multi-extension, then there is a condition q that extends p and each w_i. From this, strong properness will follow by the simple observation that a single condition $w \in \mathbb{P}_\delta \cap M^\delta_\alpha$ extending $[p]_{M^\delta_\alpha}$ is a p-multi-extension, given that p is super-nice with respect to M^δ_α:

Observation 6.2.15. *If $p \in \mathbb{P}_\delta$ is super-nice with respect to M^δ_α and $w \in \mathbb{P}_\delta \cap M^\delta_\alpha$ extends $[p]^\delta_\alpha$, then the sequence of length one with $w_0 = w$ is a p-multi-extension with respect to the δ-sequence of length one, $(\delta_0, \alpha_0) = (\delta, \alpha)$.*

We need one more lemma before proving that it indeed is possible to find common extensions for multi-extensions. Recall the dense countably closed set of *nice* conditions in $\mathrm{Col}(\omega_1, < \kappa)$ from item (6) of Proposition 4.2.1. We say that $p \in \mathbb{P}_\delta$ is **nice** if f_0^p is nice with respect to every $t \in T$. Since the set of nice conditions is dense and countabley closed, the set of conditions that are both nice and super-nice with respect to any countably many models is also dense and countably closed. We have the analogue of the flexibility property of $\dot{T}$, i.e. Proposition 4.2.1(6):

Lemma 6.2.16 (Flexibility Lemma). *Let $\delta < \kappa^+$. Suppose that $p \in \mathbb{P}_\delta$ and $t_n, n < \omega$, are countably nodes in $T - V_\alpha$ with $\ell(t_n) \leqslant \alpha$ for each $n < \omega$, and $\bar{t}_n, n < \omega$, are countably many nodes in $T \cap V_\alpha$ such that $\bar{t}_n \in \mathrm{wd}(t_n) \times \mathrm{ht}(t_n)$. Then for any $\sigma : \omega \to 2$ there is a condition $q \leqslant p$ such that*

1. $[q]_{M_\alpha^\delta} = [p]_{M_\alpha^\delta}$,
2. *if p is super-nice with respect to M_α^δ, then so is q,*
3. *any extension of q that forces $\dot{\pi}_p(t_n) \leqslant \bar{t}_n$ must also force*

$$\bar{t}_n <_{\dot{T}}^{\sigma(i)} t_n,$$

where $<_{\dot{T}}^0$ is $\not<_{\dot{T}}$ and $<_{\dot{T}}^1$ is $<_{\dot{T}}$, for every $n < \omega$.

Proof. Look at the collapse condition $f_0^p \in \mathrm{Col}(\omega, < \kappa)$. For each $n < \omega$, let $\bar{t}'_n$ be either $\bar{t}_n$ if $\sigma(n) = 1$ and otherwise let $\bar{t}'_n$ be any node in $\mathrm{wd}(t_n) \times \mathrm{ht}(t_n)$ at the height of $\bar{t}_n$ with $\bar{t}'_n \neq \bar{t}_n$. Extending ω many times, get $g \leqslant f_0^p$ be as in item (6) from Proposition 4.2.1, with respect to each t_n and $\bar{t}'_n$. Define q by letting

$$q(0) := (g, N_0^q)$$

and $q(\gamma) := p(\gamma)$ for non-zero $\gamma < \delta$. It now follows from the choice of g that any extension of q that forces ``$\pi^p(t) <_{\dot{T}} \bar{t}$" must satisfy that any common extension of $w$ and $q$ forces ``$\bar{t} <_{\dot{T}} t$". Again by choice of g we have $[p]_{M_\alpha^\delta} = [q]_{M_\alpha^\delta}$. It thus suffices to show that q is super-nice with respect to M_α^δ in case p is.

Suppose towards contradiction that p is super-nice with respect to M_α^δ but q is not. This implies that we modified the projection below a node that is relevant for super-niceness. Specifically, there is a path

$$(M_{\beta_0}^{\delta_0}, \delta_0), \ldots, (M_{\beta_n}^{\delta_n}, \delta_n)$$

in q such that $\delta_0 = \delta$ and $\beta_0 = \alpha$, and such that for some $\xi \in \delta_n \cap M^{\delta_n}_{\beta_n}$ and pair of nodes $(s,t) \in f^p_\xi$ both of which are exit nodes from $M^\xi_{\beta_n}$, we have

$$\dot\pi^q(t) \neq \dot\pi^p(t).$$

Note that the path is also in p since only the collapse condition was extended. By the minimality condition, we have $t = t_n$ for some n. In particular, $\ell(t) = \ell(t_n) \leqslant \alpha$. By definition of the poset, we have $s \in V_{\ell(t)}$. Then: $s \in M^\xi_{\ell(t)} \cap V_\kappa = V_{\ell(t)} \subseteq V_\alpha = V_{\beta_0} \subseteq V_{\beta_n}$. This contradicts the assumption that s was exit node from $M^\xi_{\beta_n}$. Hence q must be super-nice with respect to M^δ_α. ☐

We are almost ready to prove the strong version of strong properness. We need an extension of the lemma that allows to project δ-sequences (Lemma 4.1.16) into a lemma that allows to project multi-extensions.

Lemma 6.2.17. *Let $\delta < \kappa^+$ and $p \in \mathbb{P}_\delta$. Assume that $(w_j : j < \iota)$ forms a p-multi-extension with respect to $((\delta_j, \alpha_j) : j < \iota)$. For every $j < \iota$ there is $k < j$ such that $(w_i \restriction \min\{\delta_i, \delta_j\} : k \leqslant i < j)$ forms a $p \restriction \delta_j$-multi-extension with respect to $((\min\{\delta_i, \delta_j\}, \alpha_i) : k \leqslant i < j)$.*

Proof. Follows immediately from Lemma 4.1.16 by verifying the definition of multi-extension (Definition 6.2.14). ☐

We prove the strong version of strong properness.

Lemma 6.2.18 (Strong properness). *Let $\delta < \kappa^+$ and $p \in \mathbb{P}_\delta$. If $(w_j : j < \iota)$ is a p-multi-extension, then there is a condition $q \in \mathbb{P}_\delta$ that extends p and each w_j, $j < \iota$.*

Proof. The proof is by induction on δ. Let $((\delta_j, \alpha_j) : j < \iota)$ be the δ-sequence with respect to which $(w_j : j < \iota)$ is a p-multi-extension. By assumption it is an initial segment of its p-closure. Denote its p-closure by $((\delta_j, \alpha_j) : j < \iota^*)$. Denote also

$$\delta_{\iota^*} := \delta \text{ and } \alpha_{\iota^*} := \kappa.$$

Note that $\delta_j < \delta$ whenever $\iota \leqslant j < \iota^*$. Note also that it follows from definitions that $p \restriction \delta_j$ is super-nice with respect to $M^{\delta_j}_{\alpha_j}$. We proceed by recursion on $j \in [\iota, \iota^*]$ to extend the sequence $(w_j : j < \iota)$ to a sequence of length ι^*. At step $j \in [\iota, \iota^*]$, denote

$$\delta^j_i := \min\{\delta_i, \delta_j\}$$

and assume that $(w_i : i < j)$ forms a p-multi-extension with respect to the sequence $((\delta_i, \alpha_i) : i < j)$.

By Lemma 6.2.17, there is $k < j$ such that the restricted sequence $(w_i \restriction \delta_i^j : k \leqslant i < j)$ is a $p \restriction \delta_j$-multi-extension with respect to the sequence $((\delta_i^j, \alpha_i) : k \leqslant i < j)$, which by Lemma 4.1.16 is a δ_j-sequence and an element in $M_{\alpha_j}^{\delta_j}$. We define w_j in two steps. The construction happens inside the model $M_{\alpha_j}^{\delta_j}$.

Step 1. We apply Flexibility Lemma 6.2.16 to the trace $[p \restriction \delta_j]_{M_{\alpha_j}^{\delta_j}}$ and obtain $p_1 \in \mathbb{P}_{\delta_j} \cap M_{\alpha_j}^{\delta_j}$ such that:

1. $p_1 \leqslant [p \restriction \delta_j]_{M_{\alpha_j}^{\delta_j}}$,
2. p_1 is super-nice with respect to $M_{\alpha_i}^{\delta_i^j}$ for every $i \in [k, j)$ and satisfies
$$[p_1]_{M_{\alpha_i}^{\delta_i^j}} = [p \restriction \delta_j]_{M_{\alpha_i}^{\delta_i^j}}.$$
3. If $j' < j$ and $(s, t) \in f_{\delta_{j'}}^p \cap V_{\alpha_j}$, then for any
$$(s', t') \in \bigcup\{f_{\delta_{j'}}^{w_i} : i < j \text{ and } \delta_{j'} < \delta_i\},$$
and any $\beta \in \bigcup\{N_{\delta_{j'}}^{w_i} : i < j \text{ and } \delta_{j'} < \delta_i\}$,
    (a) the collapse part $f_0^{p_1}$ forces
    $$t_{s \wedge s'} = t \wedge t',$$
    (b) there is a node $t_{\bar{s}_\beta} \in T$ at the height of $\bar{s}_\beta$ such that $f_0^{p_1}$ forces
        i. $t_{\bar{s}_\beta} <_{\dot{T}} t$,
        ii. $t_{\bar{s}_\beta}$ is an exit node from $M_\beta^{\delta_{j'}}$
        iii. $\mathsf{wd}(t_{\bar{s}_\beta}) = \mathsf{wd}(\bar{s}_\beta)$.

Step 2. This step breaks into two cases: either $j < \iota^*$ or $j = \iota^*$.

Suppose first that $j < \iota^*$, and thus $\delta_j < \delta$. We observe that the sequence $(w_i \restriction \delta_i^j : k \leqslant i < j)$ forms a p_1-multi-extension with respect to the δ_j-sequence $((\delta_i^j, \alpha_i) : k \leqslant i < j)$. Both sequences, as well as the condition p_1, are elements in $M_{\alpha_i}^{\delta_j}$. By induction hypothesis applied inside the model $M_{\alpha_j}^{\delta_j}$, we obtain a condition $w' \in \mathbb{P}_{\delta_j} \cap M_{\alpha_j}^{\delta_j}$ that extends p_1 as well as each w_i, $i \in [k, j)$. Still working inside $M^{\delta_j}_{\alpha_j}$, extend $w_j \leqslant w'$ such that for every $s \in \mathsf{dom}(f^p_{\delta_j}) \cap V_{\alpha_j}$, the condition w_j decides meets in the set

$$\{s\} \cup \bigcup\{\mathsf{dom}(f^{w_i}_{\delta_j}) : i < j \text{ and } \delta_j < \delta_i\},$$

their implicit images, and for every $\beta \in \bigcup\{N^{w_i}_{\delta_j} : i < j \text{ and } \delta_j < \delta_i\}$, the exit node $\bar{s}_\beta$. This ends the definition of w_j.

Suppose then that $j = \iota^*$, and thus $\delta_j = \delta$. First let w' be the pointwise union of the conditions $(w_j : j < \iota^*)$, i.e.

$$w'(\gamma) := \left(\bigcup_{j<\iota^*} f^{w_j}_\gamma, \bigcup_{j<\iota^*} N^{w_j}_\gamma \right).$$

It follows that w' is a condition in $\mathbb{P}_\delta$, since $(w_j : j < \iota^*)$ is a descending chain. We now amalgamate it with p, by letting w_{ι^*} to be defined by letting

$$\begin{aligned} f^{w_{\iota^*}}_\gamma := f^p_\gamma \cup f^{w'}_\gamma \cup \{(s \wedge s', t_{s\wedge s'}) : s' \in \bigcup_{j<\iota^*} \mathsf{dom}(f^{w_j}_\gamma)\} \\ \cup \{(\bar{s}_\beta, t_{\bar{s}_\beta}) : \beta \in \bigcup_{j<\iota^*} N^{w_j}_\gamma\}, \end{aligned}$$

and $N^{w_{\iota^*}}_\gamma := N^{w'}_\gamma \cup N^p_\gamma$. It follows from the construction that w_{ι^*} is a condition in $\mathbb{P}_\delta$ that extends p and each w_j, $j < \iota^*$.

This ends the proof. The condition w_{i^*} is the desired common extension.

□

We obtain strong properness:

Corollary 6.2.19. *Let $\delta < \kappa^+$, $\alpha \in \mathcal{E}_\delta$ and $p \in \mathbb{P}_\delta$.*

1. *If p is super-nice with respect to M^δ_α, then its trace $[p]_{M^\delta_\alpha}$ is a residue for p into M^δ_α.*
2. *If $\alpha \in E^p_\delta$, then it is strongly $(\mathbb{P}_\delta, M^\delta_\alpha)$-generic.*

Moreover, $\mathbb{P}_\delta$ is strongly proper with respect to each model in the set $\{M^\delta_\alpha : \alpha \in \mathcal{E}_\delta\}$.

Proof. The first item follows from Lemma 6.2.18 and Observation 6.2.15. The second item follows from Lemma 6.2.12: if $\alpha \in E^p_\delta$, there is $q \leqslant p$ that is super-nice with respect to M^δ_α and thus the trace $[q]_{M^\delta_\alpha}$ is a residue for q and therefore for p too, into M^δ_α. The fourth item follows from Lemma 6.2.2: if $p \in \mathbb{P}_\delta \cap M^\delta_\alpha$, there is $q \leqslant p$ with $\alpha \in E^q_\delta$, and by the second item q is strongly $(\mathbb{P}_\delta, M^\delta_\alpha)$-generic. $\square$

Corollary 6.2.20. *In $V^{\mathbb{P}_{\kappa^+}}$, we have*

1. $\kappa = \aleph_2$,
2. $2^{\aleph_1} = \kappa^+ = \aleph_3$.

There remains to show that every $\dot{S}_\delta$ is embedded into $\dot{T}$, and that $\dot{T}$ remains Aronszajn throughout the iteration, given that none of the wide κ-trees $\dot{S}_\delta$ has a long branch.

6.2.3 Node density

We show that the embeddings

$$\dot{f}^G_\delta = \bigcup_{p \in G} f^p_\delta : \dot{S}^G_\delta \to \dot{T}^G$$

where $G \subseteq \mathbb{P}_{\kappa^+}$ is a generic filter, are defined on the whole tree $\dot{S}^G_\delta$. This follows from the following node density lemma.

Lemma 6.2.21 (Node Density). *For any $\delta < \kappa^+$ and $s \in S_\delta$, the set $\{p : s \in \mathrm{dom}(f^p_\delta)\}$ is dense in $\mathbb{P}_{\kappa^+}$.*

Proof. Up to extending p if necessary, we may assume that it decides meets of the form $s \wedge s'$ where $s' \in \mathrm{dom}(f^p_\delta)$, their **implicit images**

$$t_{s\wedge s'} := \text{the unique predecessor of } f^p_\delta(s') \text{ at the height of } s \wedge s',$$

as well as for every $\alpha \in N^p_\delta$ such that $s \notin M^\delta_\alpha$, the node

$$\bar{s}_\alpha := \text{ the predecessor of } s \text{ that is an exit node from } M^\delta_\alpha,$$

as well as its width $\mathrm{wd}(\bar{s}_\alpha)$. By Lemma 6.2.3, we may assume that the pairs $(s \wedge s', t_{s\wedge s'})$ are already in f^p_δ.

The proof is easy if $N^p_\delta = \varnothing$ or if $s \in M^\delta_\alpha$ for every $\alpha \in N^p_\delta$: in this case it suffices to select a node $t \in T$ at the height of s such that $t \in M^\delta_\alpha$ for every $\alpha \in N^p_\delta$

and such that f_0^p does not decide anything about it, i.e. $\dot{\pi}^p(t)$ is the root of the tree. Then we may extend just the collapse condition of $p \restriction \delta$ to obtain a condition p' that forces ``$t \wedge f_\delta^p(s') = t_{s \wedge s'}$", for any $s' \in \mathsf{dom}(f_\delta^p)$. This extension can be done for instance by Flexibility Lemma 6.2.16 or by item (6) of Proposition 4.2.1. Then $q := p'^\frown(f_\delta^p \cup \{(s,t)\}, N_\delta^p)^\frown p \restriction [\delta+1, \kappa^+)$ is as wanted.

We now assume that the set of $\alpha \in N_\delta^p$ such that $s \notin M_\alpha^\delta$ is non-empty. We will find images $t_{\bar{s}_\alpha}$ for each $\bar{s}_\alpha$ as well as an image t_s for s. We need to make sure that each $\bar{s}_\alpha$ is mapped to an exit node from M_α^δ, and that for any $\beta \in N_\delta^p$ with $\alpha < \beta$,

$$\bar{s}_\alpha \in M_\beta^\delta \iff t_{\bar{s}_\alpha} \in M_\beta^\delta,$$

and $s \in M_\beta^\delta$ iff $t_s \in M_\beta^\delta$. Note that it might happen that $\bar{s}_\alpha = \bar{s}_\beta$ for $\alpha < \beta$. Let $\bar{S} := \{\bar{s}_\alpha : \alpha \in N_\delta^p\} \cup \{s\}$. For each $\bar{s} \in \bar{S}$, let $\beta_{\bar{s}}$ be the least member in $N_\delta^p \cup \{\kappa\}$ such that $\bar{s} \in M_{\beta_{\bar{s}}}^\delta$. Up to extending $p \restriction \delta$ further, we may assume that it is super-nice with respect to M_α^δ for every $\alpha \in N_\delta^p$, using Lemma 6.2.12 and the observation that $N_\delta^p \subseteq E_\delta^p$. Let $((\delta_j, \alpha_j) : j < \iota)$ be the $p \restriction \delta$-closure of the set of those pairs (δ, α) such that $\alpha \in N_\delta^p$ and $s \notin M_\alpha^\delta$, and denote $(\delta_\iota, \alpha_\iota) := (\delta, \kappa)$. Note that for every $\bar{s} \in \bar{S}$ there is $j \leqslant \iota$ such that $(\delta, \beta_{\bar{s}}) = (\delta_j, \alpha_j)$.

We proceed by recursion on $j \leqslant \iota$ to define a $p \restriction \delta$-multi-extension $(w_j : j < \iota)$ with respect to $((\delta_j, \alpha_j) : j < \iota)$.

Let $w_0 := [p \restriction \delta_0]_{M_{\alpha_0}^{\delta_0}}$. At step j, assume that the conditions w_i, $i < j$, have been defined and form a $p \restriction \delta$-multi-extension with respect to the δ-sequence $((\delta_i, \alpha_i) : i < j)$. We work in the model $M_{\alpha_j}^{\delta_j}$. Denote $\delta_i^j := \min\{\delta_i, \delta_j\}$. Using Lemma 6.2.17, select $k < j$ such that

$$(w_i \restriction \delta_i^j : k \leqslant i < j)$$

is a $[p \restriction \delta_j]_{M_{\alpha_j}^{\delta_j}}$-multi-extension. By Lemma 6.2.18 applied in $M_{\alpha_j}^{\delta_j}$ there is a condition

$$w' \in \mathbb{P}_{\delta_j} \cap M_{\alpha_j}^{\delta_j}$$

that extends $[p \restriction \delta_j]_{M_{\alpha_j}^{\delta_j}}$ and each $w_i \restriction \delta_i^j$. We split into two cases.

Case 1: There is no $\bar{s} \in \bar{S}$ such that $(\delta, \beta_{\bar{s}}) = (\delta_j, \alpha_j)$. In this case we let $w_j := w'$.

Case 2: There is $\bar{s} \in \bar{S}$ such that $(\delta, \beta_{\bar{s}}) = (\delta_j, \alpha_j)$. In this case each such $\bar{s}$ that has form $\bar{s}_\alpha$ is unique, but might satisfy $\bar{s} = \bar{s}_\alpha = \bar{s}_{\alpha'}$ for multiple different $\alpha < \alpha'$ from N_δ^p. In addition to $\bar{s}_\alpha$, we might have $\bar{s} = s$. We find an image for each such $\bar{s}$. Since $\alpha_j \in N_\delta^p$, we have that α_j is a limit point of $\mathcal{E}_\delta$. Thus, since $w' \in M_{\alpha_j}^\delta$, there is

$\beta \in \mathcal{E}_\delta \cap \alpha_j$ such that $w' \in M^\delta_\beta$. By Lemma 6.2.2 applied in $M^\delta_{\alpha_j}$, there is $w'' \leqslant w'$ such that $\beta \in E^{w''}_\delta$, i.e. $\beta \in N^{w''}_\gamma$ for every $\gamma \in M^\delta_\beta$. Now, still working in the model $M^\delta_{\alpha_j}$, find a node $t_{\bar{s}} \in T$ with the following properties

- $\ell(t_{\bar{s}}) = \beta$,
- $\mathsf{wd}(t_{\bar{s}}) = \mathsf{wd}(\bar{s})$,
- $\mathsf{ht}(t_{\bar{s}}) = \mathsf{ht}(\bar{s})$,
- $t_{\bar{s}}$ is fresh in the sense that $f^{w''}_0$ nor f^p_0 do not decide anything about its place in the tree order of $\dot{T}$.

There is an extension of w'' (still in $M^\delta_{\alpha_j}$) that was obtained only by modifying the collapse part of w'', using item (6) of Proposition 4.2.1, that forces the following information about $t_{\bar{s}}$:

- ``$t_{\bar{s}_\alpha} \leqslant_{\dot{T}} t_{\bar{s}}$" for every $\alpha \in N^p_\delta \cap \alpha_j$,
- ``$t_{s \wedge s'} = t_{\bar{s}} \wedge f^p_\delta(s')$" for any $s' \in \mathsf{dom}(f^p_\delta)$.

We let w_j be this extension. Then $w_j \in \mathbb{P}_{\delta_j} \cap M^{\delta_j}_{\alpha_j}$ and $w_j \leqslant [p \restriction \delta]_{M^{\delta_j}_{\alpha_j}}$.

Finally, we define

$$f := f^p_\delta \cup \{(\bar{s}, t_{\bar{s}}) : \bar{s} \in \bar{S}\},$$

and let $q := w_{\iota^*}{}^\frown(f, N^p_\delta)^\frown p \restriction [\delta + 1, \kappa^+)$. Then q is a condition in $\mathbb{P}_{\kappa^+}$ that extends p and satisfies $s \in \mathsf{dom}(f^q_\delta)$.

□

6.2.4 Strong properness in quotients

We end the section with considerations about strong properness in the quotients by models M^δ_α. The lemmas of this subsection will be needed in order to prove that the iteration does not create a long branch through $\dot{T}$, which will be done in the next section.

We begin by revising the notion of super-nice condition. Let $\gamma < \delta < \kappa^+$ and $\alpha \in \mathcal{E}_\delta$, and suppose that p is a condition in $\mathbb{P}_\gamma$ that satisfies $\alpha \in N^p_\xi$ for every $\xi \in \gamma \cap M^\delta_\alpha$.

We can lift p, which is a condition in $\mathbb{P}_\gamma$, naturally to a condition q in $\mathbb{P}_\delta$ that satisfies $\alpha \in E^q_\delta$, by adding α to N^q_ξ for every $\xi \in (\delta - \gamma) \cap M^\delta_\alpha$. The condition q obtained this way is a condition in $\mathbb{P}_\delta$ that satisfies $\alpha \in E^q_\delta$ and $q \restriction \delta = p$.

Definition 6.2.22. Let $\gamma \leqslant \delta$ and let $\alpha \in \mathcal{E}_\delta$. A condition $p \in \mathbb{P}_\gamma$ is **super-nice with respect to M^δ_α** if $\alpha \in N^q_\xi$ for every $\xi \in \gamma \cap M^\delta_\alpha$ and the condition $q \in \mathbb{P}_\delta$ obtained from p by letting

$$q(\xi) := \begin{cases} p(\xi) & \text{if } \xi < \gamma, \\ (\varnothing, \{\alpha\}) & \text{if } \xi \in (\delta - \gamma) \cap M^\delta_\alpha \\ (\varnothing, \varnothing) & \text{if } \xi \in (\delta - \gamma) - M^\delta_\alpha \end{cases}$$

is super-nice with respect to M^δ_α.

Recall the trace operator $[p]_M$ from Definition 6.2.4. The trace $[p]_{M^\delta_\alpha}$ is defined also for conditions $p \in \mathbb{P}_\gamma$ and models M^δ_α, where possibly $\gamma < \delta$. It works as expected in the case of this generalized version of super-niceness:

Lemma 6.2.23. *Let $\gamma \leqslant \delta$ and let $\alpha < \kappa$. If $p \in \mathbb{P}_\gamma$ is super-nice with respect to M^δ_α, then:*

1. *the conditions that are super-nice with respect to M^δ_α are dense below p in $\mathbb{P}_\gamma$,*
2. *the trace $[p]_{M^\delta_\alpha}$ is a condition in $\mathbb{P}_\gamma \cap M^\delta_\alpha$ and a residue of p into M^δ_α.*

Proof. Follows from Lemmas 6.2.12 and 6.2.18 by identifying conditions in $\mathbb{P}_\gamma$ with conditions in $\mathbb{P}_\delta$ along the map $p \mapsto p^\frown((\varnothing, \varnothing), \dots)$. $\square$

It follows that if $p \in \mathbb{P}_\gamma$ is super-nice with respect to M^δ_α, then every generic $G \subseteq \mathbb{P}_\gamma \cap M^\delta_\alpha$ that contains $[p]_{M^\delta_\alpha}$ extends to a generic on $\mathbb{P}_\gamma$ that contains p.

Lemma 6.2.24. *Let $\gamma \leqslant \delta < \kappa^+$, $\alpha \in \mathcal{E}_\delta$ and $\beta \in \mathcal{E}_\gamma$. If there is a path from M^δ_α to M^γ_β (see Definition 4.1.8), then the trace map $[\cdot]_{M^\delta_\alpha}$ on $\mathbb{P}_\gamma \cap M^\gamma_\beta$ is a residue map into M^δ_α on conditions that are super-nice with respect to M^δ_α and element in M^γ_β.*

Proof. Note that if there is a path from M^δ_α to M^γ_β, then $\gamma \cap M^\delta_\alpha \subseteq M^\gamma_\beta$ by Lemma 4.1.9. Thus, if $p \in \mathbb{P}_\gamma \cap M^\gamma_\beta$ is super-nice with respect to M^δ_α and $w \in \mathbb{P}_\gamma \cap M^\delta_\alpha$ extends $[p]_{M^\delta_\alpha}$, then $w \in M^\gamma_\beta$, and thus by elementarity w and p have a common extension in $\mathbb{P}_\gamma \cap M^\gamma_\beta$. $\square$

If $G \subseteq \mathbb{P}_\delta$ and $\gamma < \delta$, we denote $\boldsymbol{G \restriction \gamma} := \{p \restriction \gamma : p \in G\}$.

Lemma 6.2.25. *Let $\gamma \leqslant \delta < \kappa^+$ and $\alpha \in \mathcal{E}_\delta$. Suppose that $p \in \mathbb{P}_\gamma$ is super-nice with respect to M^δ_α. Let $G \subseteq \mathbb{P}_\gamma \cap M^\delta_\alpha$ be a generic filter and let $\beta \in \mathcal{E}_\gamma - (\alpha + 1)$ be such that there is a path from M^δ_α to M^γ_β.*

1. *If $p \in (\mathbb{P}_\gamma/G) \cap M^\delta_\beta$, then there is $q \leqslant p$ in $\mathbb{P}_\gamma/G$ that satisfies $\beta \in N^q_\xi$ for every $\xi \in \gamma \cap M^\delta_\beta$.*
2. *If $p \in \mathbb{P}_\gamma/G$ satisfies $\beta \in N^p_\xi$ for every $\xi \in \gamma \cap M^\delta_\beta$, then there is $q \leqslant p$ in $\mathbb{P}_\gamma/G$ that is super-nice with respect to M^δ_β.*
3. *If D is a dense and countably closed subset of $\mathbb{P}_\gamma$, then $D \cap (\mathbb{P}_\gamma/G)$ is dense and countably closed subset of $\mathbb{P}_\gamma/G$.*
4. *If $(w_j : j < \iota)$ is a p-multi-extension with respect to a γ-sequence $((\delta_j, \alpha_j) : j < \iota)$ such that $w_j \in \mathbb{P}_{\delta_j}/G \restriction \delta_j$ for every $j < \iota$, then there is a common extension of p and each w_j in $\mathbb{P}_\gamma/G$.*

Proof. Items (1), and (2) are proved exactly like Lemmas 6.2.2 and 6.2.12. For item (3): First density. Let $p \in \mathbb{P}_\gamma/G$. If to the contrary there is no extension of p in $D \cap (\mathbb{P}_\gamma/G)$, then there is a condition $w \in G$ that forces this. Up to exteding w, we may assume that it extends a residue of p. But this is not possible since there is an extension of $q \leqslant w, p$ that is super-nice with respect to M^δ_α and can be assumed to be below a condition in D. Then $[q]_{M^\gamma_\alpha}$ extends w and forces ``$q \in D \cap (\mathbb{P}_\gamma/G)$". To see countable closure, suppose that $(p_n : n < \omega)$ is a descending sequence in $D \cap (\mathbb{P}_\gamma/G)$. We may assume that $r \in G$ forces ``$p_n \in D \cap (\mathbb{P}_\gamma/\check{G})$" for each $n < \omega$, and to the contrary, that ``$D \cap (\mathbb{P}_\gamma/\check{G})$ does not contain a lower bound for the $p_n$, $n < \omega$". We again obtain a contradiction: find $q$ that is super-nice and extends $r$ and each $p_n$. Then $[q]_{M^\delta_\alpha} \leqslant r$ and $[q]_{M^\delta_\alpha}$ forces ``$q \in D \cap (\mathbb{P}_\gamma/\check{G})$ and q is a lower bound for the p_n, $n < \omega$". This is a contradiction. We have proved item (3). Item (4) is proved similarly, using Lemma 6.2.18. □

6.3 Preservation of Aronszajnness

In this section, we prove that if all chosen names $\dot{S}_\delta$ are names for wide Aronszajn trees, then $\dot{T}$ does not get a cofinal branch, and so can become a universal wide Aronszajn tree on $\kappa = \aleph_2$.

The following definition is inspired by [16].

Definition 6.3.1. Let $\delta < \kappa^+$ and let $\dot{S}$ be a $\mathbb{P}_\delta$-name for a tree. Two conditions $p, q \in \mathbb{P}_\delta$ are said to **split a pair of nodes (s, s')** from $\dot{S}$ if there are distinct nodes $\bar{s} \neq \bar{s}'$ at some level $\bar{\alpha}$ such that

1. $p \Vdash \bar{s} <_{\dot{S}} s$
2. $q \Vdash \bar{s}' <_{\dot{S}} s'$

We say that p and q split the node s if they split the pair (s, s).

Note that in the above definition, the case $s = s'$ is not excluded.

Lemma 6.3.2. *Let $\delta < \kappa^+$ and $\alpha \in \mathcal{E}_\delta$. If $G \subseteq \mathbb{P}_\delta \cap M^\delta_\alpha$ is a generic filter and if $s \in S_\delta$ is an exit node from M^δ_α at a limit level, then the branch below s is not introduced by G.*

Proof. Suppose towards contradiction that $b \subseteq (\dot{S}_\delta \cap V_\alpha)^G$ is forced to be the branch below s. By the Π^1_1-reflection, the tree $(\dot{S}_\delta \cap V_\alpha)^G$ is a wide α-Aronszajn tree (See Observation 2.2.4). If s has height α, then the branch below it cannot be introduced by G, for it would be a cofinal branch in $(\dot{S}_\delta \cap V_\alpha)^G$. Thus the height $\bar{\alpha}$ of s must satisfy $\bar{\alpha} < \alpha$. Consider $\bar{\beta} :=$ the minimal ordinal such that $b \subseteq \bar{\beta} \times \bar{\alpha}$. We have $\bar{\beta} \leqslant \alpha$ because s is an exit node from M^δ_α. If $\bar{\beta} = \alpha$, then b induces a cofinal function from $\bar{\alpha}$ to α, which is absurd since $\alpha = \aleph_2^{V[G]}$. Thus $\bar{\beta} < \alpha$. Let G' be a generic on $\mathbb{P}_\delta$ that extends G. It follows from Lemma 4.1.7 that the set $M^\delta_\alpha[G'] = \{\dot{a}^{G'} : \dot{a} \in V^{\mathbb{P}_\delta} \cap M^\delta_\alpha\}$ is an elementary submodel of $H_{\kappa^{++}}[G']$ and closed under $<\alpha$-sequences. Thus $b \in M^\delta_\alpha[G']$. And by elementarity, b must have a supremum in $M^\delta_\alpha[G']$. Since s is at a limit level, it is the unique supremum of b. This is absurd since s is an exit node from $M^\delta_\alpha[G']$. □

Lemma 6.3.3. *Let $\delta < \kappa^+$. For every $p \in \mathbb{P}_\delta$ that is super-nice with respect to M^δ_α there is $q \leqslant p$ that is super-nice with respect to M^δ_α and satisfies that if $((\delta_j, \alpha_j) : j < \iota)$ is the q-closure of M^δ_α, then*

$$\sup\{\alpha_j : j < \iota\} = \sup\{\alpha_j : j < \iota \textit{ and } \delta_j \in \delta \cap M^\delta_\alpha\}.$$

Moreover, this supremum can be chosen as high in κ as wanted.

Proof. By adding nodes to coordinates in M^δ_α using Node Density Lemma 6.2.21. □

Definition 6.3.4. Let $\gamma \leqslant \delta$ and $\alpha \in \mathcal{E}_\delta$. A pair of conditions (p, q) from $\mathbb{P}_\gamma$ **nicely splits with respect to M^δ_α** if

1. $\alpha \in N^p_\xi \cap N^q_\xi$ for every $\xi \in \gamma \cap M^\delta_\alpha$,
2. for every $\xi \in \gamma \cap M^\delta_\alpha$ and every pair $(s, s') \in \mathsf{dom}(f^p_\xi) \times \mathsf{dom}(f^q_\xi)$ of exit nodes from M^ξ_α at some limit levels, the conditions $p \restriction \xi$ and $q \restriction \xi$ split the pair (s, s') with a pair $(\bar{s}, \bar{s}') \in (\mathsf{dom}(f^p_\xi) \times \mathsf{dom}(f^q_\xi)) \cap M^\xi_\alpha$.
3. for every $\xi \in \gamma \cap M^\delta_\alpha$ and every $s \in \mathsf{dom}(f^p_\xi)$ (resp. $s \in \mathsf{dom}(f^q_\xi)$) exit node from M^ξ_α at a successor level, the condition $p \restriction \xi$ (resp. $q \restriction \xi$) decides the immediate predecessor of s and it belongs to $\mathsf{dom}(f^p_\xi)$ (resp. to $\mathsf{dom}(f^q_\xi)$).

Lemma 6.3.5. *Let $\gamma \leqslant \delta < \kappa^+$, $p, q \in \mathbb{P}_\gamma$ and $\alpha \in \mathcal{E}_\delta$.*

1. *If p and q nicely split with respect to M^δ_α and $[p]_{M^\delta_\alpha} = [q]_{M^\delta_\alpha}$ then there are $\hat{p} \leqslant p$ and $\hat{q} \leqslant q$ that are super-nice with respect to M^δ_α and $[\hat{p}]_{M^\delta_\alpha} = [\hat{q}]_{M^\delta_\alpha}$. Moreover, for any $\beta \in \mathcal{E}_\gamma$ it holds that if $p \in M^\gamma_\beta$, then $\hat{p} \in M^\gamma_\beta$.*
2. *If p and q are super-nice with respect to M^δ_α and have the same trace to it, and $t \in T$ is an exit node from V_α at a limit level, then there are $\hat{p} \leqslant p$ and $\hat{q} \leqslant q$ in $\mathbb{P}_\gamma$ that nicely split with respect to M^δ_α and $[p]_{M^\delta_\alpha} = [q]_{M^\delta_\alpha}$, and furthermore split t. Moreover, for any β it holds that if $p \in M^\gamma_\beta$, then $\hat{p} \in M^\gamma_\beta$.*
3. *$\dot{T}$ is a wide κ-Aronszajn tree in $V^{\mathbb{P}_\gamma}$.*

Proof. The proof is by induction on $\gamma < \kappa^+$. We begin with a remark.

Remark 6.3.6. If the first two items of Lemma 6.3.5 hold for γ, then any pair of conditions $p, q \in \mathbb{P}_\gamma$ having the same trace to M^δ_α satisfies that if either they nicely split or are super-nice with respect to M^δ_α, then there are extensions $\hat{p} \leqslant p$ and $\hat{q} \leqslant q$ that still have the same trace to M^δ_α and both nicely split and are super-nice with respect to M^δ_α. This can be proved by iterating ω many times items (1) and (2). And moreover, $\hat{p}$ can be guaranteed to be a member of $M^\gamma_{\hat{\beta}}$ where $\hat{\beta}$ is the least such that $p \in M^\gamma_{\hat{\beta}}$.

Proof of item 1.

We look at the successor case $\gamma + 1$ first. Let $p, q \in \mathbb{P}_{\gamma+1}$ be such that they nicely split with respect to M^δ_α and satisfy $[p]_{M^\delta_\alpha} = [q]_{M^\delta_\alpha}$, for some $\delta \geqslant \gamma + 1$ and $\alpha \in \mathcal{E}_\delta$. In order to make sure that the 'Moreover'-part will be satisfied, let $\hat{\beta}$ be the least member of $\mathcal{E}_{\gamma+1}$ such that $p \in M^{\gamma+1}_{\hat{\beta}}$. By induction hypothesis and Remark 6.3.6 we may assume that $p \restriction \gamma$ and $q \restriction \gamma$ both nicely split and are super-nice with respect to M^δ_α.

In case $\gamma \notin M^\delta_\alpha$ we are done, since in this case $[p]_{M^\delta_\alpha}(\gamma) = \varnothing = [q]_{M^\delta_\alpha}(\gamma)$, and this together with the fact that $p \restriction \gamma$ and $q \restriction \gamma$ are super-nice with respect to M^δ_α and have the same trace to it guarantees that also p and q are super-nice with respect to M^δ_α and have the same trace to it.

Thus we assume $\gamma \in M^\delta_\alpha$.

Let us momentarily fix a generic $G \subseteq \mathbb{P}_\gamma \cap M^\delta_\alpha$ that contains the common trace of $p \restriction \gamma$ and $q \restriction \gamma$ into M^δ_α. Then by Lemma 6.2.23 we have $p \restriction \gamma, q \restriction \gamma \in \mathbb{P}_\gamma/G$. In case $\gamma \in M^\delta_\alpha$, the function $(f^p_\gamma \cup f^q_\gamma) \cap M^\delta_\alpha$ is a meet- and level-preserving tree-embedding from $(\dot{S}_\gamma \cap V_\alpha)^G$ to $(\dot{T} \cap V_\alpha)^G$. In the quotient $\mathbb{P}_\gamma/G$, by dovetailing, using induction hypothesis and an appropriate bookkeeping function that enumerates the relevant exit nodes from $\mathsf{dom}(f^p_\gamma)$, by iterating ω many times as in the proof of super-nice density (Lemma 6.2.12), we find $p_0 \in M^\gamma_{\hat{\beta}} \cap \mathbb{P}_\gamma/G$ extending $p \restriction \gamma$ such that

a. p_0 is super-nice with respect to every M^γ_β where $\beta \in E^p_\gamma$,

b. if $\gamma \in M^\delta_\alpha$, then p_0 decides, for every pair $(s,t) \in f^p_\gamma$ of exit nodes from V_α, cofinal implicit preimages for the node projection $\dot{\pi}^{p_0}(t)$, i.e. there are nodes $(\bar{t}^p_n : n < \omega)$ cofinal in $\dot{\pi}^{p_0}(t)$ and p_0 decides, for each n, the node $\bar{s}^p_n$ below s at the height of $\bar{t}^p_n$.

Then, similarly, find $q_0 \in \mathbb{P}_\gamma/G$ extending $q \restriction \gamma$ such that

c. q_0 is super-nice with respect to every M^γ_β where $\beta \in E^p_\gamma$,

d. if $\gamma \in M^\delta_\alpha$, then same for q_0: q_0 decides, for every pair $(s,t) \in f^p_\gamma$ of exit nodes from V_α, cofinal implicit preimages for the node projection $\dot{\pi}^{q_0}(t)$, i.e. there are nodes $(\bar{t}^q_n : n < \omega)$ cofinal in $\dot{\pi}^{q_0}(t)$ and q_0 decides, for each n, the node $\bar{s}^q_n$ below s at the height of $\bar{t}^q_n$.

Since the conditions p_0 and q_0 are in the quotient $\mathbb{P}_\gamma/G$, we may assume that they are super-nice with respect to M^δ_α and have the same trace to it, and moreover $p_1 \in M^\gamma_{\hat{\beta}}$. We now forget the generic G. By induction hypothesis for γ, using Remark 6.3.6, we find $p_1 \leqslant p_0$ and $q_1 \leqslant q_0$ in $\mathbb{P}_\gamma$ that nicely split, are super-nice with respect to M^δ_α and have same trace to it. Repeating the construction above, we find $p_2 \leqslant p_1$ and $q_2 \leqslant q_1$ that will take care of the super-niceness of the functions f^p_γ and f^q_γ – in other words by momentarily working in the quotient of $\mathbb{P}_\gamma$ by a generic on $\mathbb{P}_\gamma \cap M^\delta_\alpha$ that contains the common trace $[p_1]_{M^\delta_\alpha} = [q_1]_{M^\delta_\alpha}$, we find $p_2 \leqslant p_1$ and $q_2 \leqslant q_1$ that are super-nice with respect to M^δ_α, have same trace to it, and furthermore decide the relevant "implicit preimages for node-projections" from items b. and d. from above. We continue this recursion ω many times. At odd steps, we take care of items b. and d. from above and at odd steps, we make sure that the conditions nicely split. Moreover, we make sure that $p_n \in M^\gamma_{\hat\beta}$ for every n. Finally we let p_ω and q_ω be the simple amalgamations of the p_n, $n < \omega$, and the q_n, $n < \omega$, respectively.

It now holds that p_ω and q_ω are two conditions in $\mathbb{P}_\gamma$ that nicely split and are super-nice with respect to M^δ_α, and have the same trace to it, and $p_\omega \in M^\gamma_{\hat\beta}$, as $M^\gamma_{\hat\beta}$ is closed for countable sequences.

For the rest of the proof, fix again a generic filter $G \subseteq \mathbb{P}_\gamma \cap M^\delta_\alpha$ that contains the common trace of p_ω and q_ω into M^δ_α. In particular $p_\omega, q_\omega \in \mathbb{P}_\gamma/G$. For every pair $(s, s') \in \mathsf{dom}(f^p_\gamma) \times \mathsf{dom}(f^q_\gamma)$, look at the pair $(\bar{s}, \bar{s}')$ in V_α that witness the splitting, i.e. $p_\omega \Vdash \bar{s} < s$ and $q_\omega \Vdash \bar{s}' < s'$ and $\bar{s} \neq \bar{s}'$. Then the meet $\bar{s} \wedge \bar{s}'$ is decided by G because $\bar{s}, \bar{s}' \in M^\delta_\alpha$, so in particular the conditions p_ω and q_ω must decide and agree on the meet $\bar{s} \wedge \bar{s}'$. Similarly to item b. from above, denote by $\langle(\bar{s}^p_n, \bar{t}^p_n) : n < \omega\rangle$ members of the function f^p_γ such that p_ω forces that $\bar{t}^p_n$, $n < \omega$, are cofinal in the node-projection $\dot{\pi}^{p_\omega}(f^p_\gamma(s))$. Denote by $\langle(\bar{s}'^q_n, \bar{t}'^p_n) : n < \omega\rangle$ the analogous sequence for q_ω and s'. Then, both p_ω and q_ω force that for any n and m we have $\bar{s}^p_n \wedge \bar{s}'^q_m = \bar{s} \wedge \bar{s}'$ and moreover,

$$p_\omega \Vdash s \wedge \bar{s}'^q_m = \bar{s} \wedge \bar{s}',$$
$$q_\omega \Vdash \bar{s}^p_n \wedge s' = \bar{s} \wedge \bar{s}'.$$

We also have that the height of the node

$$f^p_\gamma(\bar{s}) \wedge f^q_\gamma(\bar{s}')$$

in the tree $(\dot{T} \cap V_\alpha)^G$ is exactly the height of the node $\bar{s} \wedge \bar{s}'$ in the tree $(\dot{S}_\gamma \cap V_\alpha)^G$. This follows from the fact that the trace are the same,

$$[p_\omega{}^\frown(f^p_\gamma, N^p_\gamma)]_{M^\delta_\alpha} = [q_\omega{}^\frown(f^q_\gamma, N^q_\gamma)]_{M^\delta_\alpha},$$

as follows: Since $\bar{s}, \bar{s}' \in V_\alpha$, in particular both $\bar{s}$ and $\bar{s}'$ belong to the domain of the γ-th embedding of this common trace, call it $f := f^p_\gamma \cap V_\alpha = f^q_\gamma \cap V_\alpha$. Thus, since f is meet-preserving, we have

$$f(\bar{s} \wedge \bar{s}') = f(\bar{s}) \wedge f(\bar{s}').$$

Note also that for any $n, m < \omega$, we have that both p_ω and q_ω forces

$$\bar{s}_n \wedge \bar{s}'_m = \bar{s} \wedge \bar{s}'.$$

It follows that the condition p_ω decides meets in the set

$$\begin{aligned}\mathsf{dom}(f^p_\gamma) &\cup \{\bar{s}^p_n : n<\omega, s\in \mathsf{dom}(f^p_\gamma) \text{ exit node from } V_\alpha\}\\ &\cup \{\bar{s}'^q_n : n<\omega, s'\in \mathsf{dom}(f^q_\gamma) \text{ exit node from } V_\alpha\}\end{aligned}$$

and the condition q_ω decides meets in the set

$$\begin{aligned}\mathsf{dom}(f^q_\gamma) &\cup \{\bar{s}^p_n : n<\omega, s\in \mathsf{dom}(f^p_\gamma) \text{ exit node from } V_\alpha\}\\ &\cup \{\bar{s}'^q_n : n<\omega, s'\in \mathsf{dom}(f^q_\gamma) \text{ exit node from } V_\alpha\}.\end{aligned}$$

and these meets are computed the same way by p_ω and q_ω and already in the domain of the function f. Hence we may let

$$\begin{aligned}\hat{f}^p &:= f^p_\gamma \cup \{(\bar{s}^p_n, \bar{t}^p_n) : (s,t)\in f^p_\gamma\}\\ &\qquad \cup \{(\bar{s}'^q_m, \bar{t}'^q_m) : (s',t')\in f^q_\gamma\}\\ \hat{f}^q &:= f^q_\gamma \cup \{(\bar{s}^p_n, \bar{t}^p_n) : (s,t)\in f^p_\gamma\}\\ &\qquad \cup \{(\bar{s}'^q_m, \bar{t}'^q_m) : (s',t')\in f^q_\gamma\}.\end{aligned}$$

It follows from above that both $\hat{f}^p$ and $\hat{f}^q$ are injective functions, p_ω forces that the domain of the function $\hat{f}^p$ is closed under meets and is level- and meet-preserving, and q_ω forces that the function $\hat{f}^q$ is closed under meets and is level- and meet-preserving. It also follows that

$$\begin{aligned}\hat{p} &:= p_\omega{}^\frown(\hat{f}^p, N^p_\gamma),\\ \hat{q} &:= q_\omega{}^\frown(\hat{f}^q, N^q_\gamma)\end{aligned}$$

are conditions in $\mathbb{P}_{\gamma+1}$ that are super-nice with respect to $M^{\gamma+1}_\alpha$ and have the same trace to it. Note that $p_\omega \in M^\gamma_{\hat{\beta}}$ and $(f^p_\gamma, N^p_\gamma)\in M^\gamma_{\hat{\beta}}$, so also $\hat{p}\in M^\gamma_{\hat{\beta}}$. Thus $\hat{p}$ and $\hat{q}$ are as wanted. This ends the successor case $\gamma+1$.

We consider the case when γ is a limit ordinal of countable cofinality. This is straightforward. Let $p,q\in\mathbb{P}_\gamma$ and suppose that they nicely split with respect to M^δ_α ans satisfy $[p]_{M^\delta_\alpha} = [q]_{M^\delta_\alpha}$. Again for the 'Moreover' -part, let $\hat{\beta}$ be the least such that $p\in M^\gamma_{\hat{\beta}}$. Fix a cofinal sequence $(\gamma_j)_{j<\omega}$ converging to γ. Without loss of generality we may assume that $\gamma_j\in M^\gamma_{\hat{\beta}}$ for every j. Proceed by recursion on $j<\omega$. Let $p_0 := p\restriction\gamma_0$ and $q_0 := q\restriction\gamma_0$. At step $j+1$, look at p_j and q_j. Suppose that they nicely split with respect to M^δ_α and satisfy $[p_j]_{M^\delta_\alpha} = [q_j]_{M^\delta_\alpha}$, and satisfy $p_j\in M^\gamma_{\hat{\beta}}$. Then $p' := p_j{}^\frown p\restriction[\gamma_j,\gamma_{j+1})$ and $q' := q_j{}^\frown q\restriction[\gamma_j,\gamma_{j+1})$ nicely split with respect to M^δ_α and satisfy $[p']_{M^\delta_\alpha} = [q']_{M^\delta_\alpha}$ By induction hypothesis there are $p_{j+1} \leqslant p'$ and $q_{j+1} \leqslant q'$ that nicely split with respect to M^δ_α and satisfy $[p_{j+1}]_{M^\delta_\alpha} = [q_{j+1}]_{M^\delta_\alpha}$, and furthermore are super-nice with respect to M^δ_α. Moreover, we may assume that $p_{j+1} \in M^\gamma_{\hat\beta}$. Finally the pointwise unions of the p_j and q_j, respectively, are as wanted.

The limit case of uncountable cofinality is a straightforward implication of induction hypothesis. This ends the proof of item 1.

Proof of item 2.

Suppose that p and q are super-nice with respect to M^δ_α and have the same trace to it. Let $t \in T$ be an exit node from M^δ_α at a limit level. We may assume without loss of generality that for every $\xi \in \gamma \cap M^\delta_\alpha$ and $s \in \mathsf{dom}(f^p_\xi)$ (resp. $s \in \mathsf{dom}(f^q_\xi)$) that is an exit node from M^δ_α at a successor level, the condition $p \restriction \xi$ (resp. $q \restriction \xi$) decides the immediate predecessor of s and it is in $\mathsf{dom}(f^p_\xi)$ (resp. in $\mathsf{dom}(f^q_\xi)$). This assumption can be made by fixing a generic $G \subseteq \mathbb{P}_\gamma \cap M^\delta_\alpha$ containing the common trace of p and q and extending p and q in the quotient $\mathbb{P}_\gamma/G$ using Node Density Lemma 6.2.21 in the quotient. Also, if $((\delta_j, \alpha_j) : j < \iota)$ is the p-closure of M^δ_α, up to extending p in the quotient $\mathbb{P}_\gamma/G$, we may assume that

$$\sup\{\alpha_j : j < \iota\} = \sup\{\alpha_j : j < \iota \text{ and } \delta_j \in M^\delta_\alpha\},$$

and moreover, that $t \in V_{\alpha_j}$ for some $j < \iota$. In particular, then there is $j < \iota$ such that $t \in V_{\alpha_j}$ and $\delta_j \in M^\delta_\alpha$. This assumption is justified by Lemma 6.3.3.

Fix thus such p and q and let $((\delta_j, \alpha_j) : j < \iota)$ be the p-closure of M^δ_α. Denote $\delta_\iota := \delta$ and $\alpha_\iota := \kappa$. Notice that $\delta_j < \gamma$ whenever $0 < j < \iota$, since $p \in \mathbb{P}_\gamma$. This is because if δ_j belongs to the p-closure of M^δ_α and is not δ, then there is a path in p to some $M^{\delta_j}_\beta$, where $\delta_j \in \sup(p) \subseteq \gamma$. By recursion on $j \leqslant \iota$, we define conditions $p_j \in \mathbb{P}_{\delta_j} \cap M^{\delta_j}_{\alpha_j}$ and conditions $q_j \in \mathbb{P}_\gamma$. The final conditions p_ι and q_ι will nicely split with respect to M^δ_α, split t, and satisfy $[p_\iota]_{M^\delta_\alpha} = [q_\iota]_{M^\delta_\alpha}$. Moreover, it will hold that if $\hat\beta$ is the least such that $p \in M^\gamma_{\hat\beta}$, then $p_\iota \in M^\gamma_{\hat\beta}$. This will be a consequence of the fact that if $p \in M^\gamma_{\hat\beta}$, then the p-closure of M^δ_α belongs to $M^\gamma_{\hat\beta}$ as well, being definable from p and α.

Let $p_0 := [p]_{M^\delta_\alpha}$ and $q_0 := q$.

At step j: Denote

$$\delta^j_i := \min\{\delta_i, \delta_j\}.$$

Assume that we have defined p_i and q_i for $i < j$ and they satisfy:

1. There is k such that $(p_i \restriction \delta_i^j : k \leqslant i < j)$ forms a $[p \restriction \delta_j]_{M_{\alpha_j}^{\delta_j}}$-multi-extension with respect to the sequence $((\delta_i^j, \alpha_i) : k \leqslant i < j)$.
2. $(q_i : i < j)$ is a decreasing sequence of conditions in $\mathbb{P}_\gamma$.
3. p_i and $q_i \restriction \delta_i$ nicely split and are super nice with respect to M_α^δ and satisfy $[p_i]_{M_\alpha^\delta} = [q_i \restriction \delta_i]_{M_\alpha^\delta}$.
4. If $\delta_i \in M_\alpha^\delta$ and $t \in V_{\alpha_i}$, then p_i and $q_i \restriction \delta_i$ split t,
5. If $\delta_i \in \gamma \cap M_\alpha^\delta$, then for every pair $(s, s') \in A_i \times B_i$, where
$$A_i = \{s \in \mathsf{dom}(f_{\delta_i}^p) : s \text{ is an exit node from } V_\alpha\} \cap (V_{\alpha_i} - \bigcup_{i'<i} V_{\alpha_{i'}}),$$
$$B_i = \{s \in \mathsf{dom}(f_{\delta_i}^{q_i}) : s \text{ is an exit node from } V_\alpha\},$$
   p_i and $q_i \restriction \delta_i$ split the pair (s, s') with a pair $(\bar{s}, \bar{s}')$. Also, the common trace $[p_i]_{M_\alpha^\delta} = [q_i \restriction \delta_i]_{M_\alpha^\delta}$ decides the meet $\bar{s} \wedge \bar{s}'$ and p_i forces
$$s \wedge \bar{s}' = \bar{s} \wedge \bar{s}'.$$
   Furthermore, $\bar{s}, \bar{s}' \in \mathsf{dom}(f_{\delta_i}^{q_i})$, and if $i' \in [i+1, j)$ is such that $f_{\delta_i}^p(s) \in V_{\alpha_{i'}}$, then $p_{i'}$ forces that
$$f_{\delta_i}^{q_i}(\bar{s}) <_{\dot{T}} f_{\delta_i}^p(s).$$
6. The extension $q_i \leqslant \bigcup_{i'<i} q_{i'}$ is minimal in the sense that we only extended the part of q_i up to δ_i and the function $f_{\delta_i}^{q_i}$ below α by adding witnesses for splitting. Specifically:
   - $q_i \restriction [\delta_i + 1, \delta) = \bigcup_{i'<i} q_{i'} \restriction [\delta_i + 1, \delta)$,
   - $N_{\delta_i}^{q_i} = \bigcup_{i'<i} N_{\delta_i}^{q_{i'}}$,
   - $f_{\delta_i}^{q_i} - V_\alpha = \bigcup_{i'<i} f_{\delta_i}^{q_{i'}} - V_\alpha$,
   - $q_i \restriction \delta_i \Vdash \mathsf{dom}(f_{\delta_i}^{q_i})$ is the least set closed under meets and exit nodes from models in $\bigcup_{i'<i} N_{\delta_{i'}}^{q_{i'}}$ that contains $\bigcup_{i'<i} \mathsf{dom}(f_{\delta_{i'}}^{q_{i'}})$ and $\{\bar{s}, \bar{s}' : (s, s') \in A_i \times B_i\}$.

Now the goal is to define p_j and q_j. This is done in multiple steps. First we will work in the model $M_{\alpha_j}^{\delta_j}$ and look at the trace $[p \restriction \delta_j]_{M_{\alpha_j}^{\delta_j}}$. We first extend its collapse part to obtain a condition $\tilde{p}$ as in the proof of strong properness, using Flexibility Lemma

6.2.16. Then we will find a condition p^0_j that extends $\tilde{p}$ and each $p_i \upharpoonright \delta^j_i$, using the strong properness for sequences, i.e. Lemma 6.2.18. We then let q^0_j to be the simple amalgamation of the q_i, $i < j$. We extend these p^0_j and q^0_j few more times to find the final conditions p_j and q_j.

We begin with finding $\tilde{p}$. The role of $\tilde{p}$ is to accommodate the splitting that was already done in the $\dot{S}_\gamma$-side to the image side $\dot{T}$. Indeed, for each pair (s, s') that has been splitted by some $(\bar{s}, \bar{s}')$ as in item 5 of the assumptions listed above, whenever $f^p_{\delta_i}(s) \in M^{\delta_j}_{\alpha_j}$, then we want $\tilde{p}$ to force ``$\check{f}_{\delta_i}(\bar{s}) < \check{f}_{\delta_i}(s)$". By assumption, the so far defined sequence $(p_i \upharpoonright \delta^j_i : i < j)$ forms a $[p \upharpoonright \delta_j]_{M^{\delta_j}_{\alpha_j}}$-multi-extension. We will find $\tilde{p} \leqslant [p \upharpoonright \delta_j]_{M^{\delta_j}_{\alpha_j}}$ in such a way that the sequence $(p_i \upharpoonright \delta^j_i : i < j)$ forms a $\tilde{p}$-multi-extension. This is possible using Lemma 6.2.16 – the condition $\tilde{p}$ is obtained only extending the collapse part of $[p \upharpoonright \delta_j]_{M^{\delta_j}_{\alpha_j}}$ above each α_i, $i < j$, at coordinates that concern nodes $t \in T$ with $\ell(t) \leqslant \sup_{i<j} \alpha_i$. This guarantees that $\tilde{p}$ is super-nice with respect to every $M^{\delta^j_i}_{\alpha_i}$, just like the trace $[p \upharpoonright \delta_j]_{M^{\delta_j}_{\alpha_j}}$, and has the same trace with it to each smaller model, i.e. $[\tilde{p}]_{M^{\delta^j_i}_{\alpha_i}} = [[p \upharpoonright \delta_j]_{M^{\delta_j}_{\alpha_j}}]_{M^{\delta^j_i}_{\alpha_i}} = [p]_{M^{\delta^j_i}_{\alpha_i}}$. In particular, we intend the future extension of $\tilde{p}$ to satisfy the ‘Furthermore’ -part of item (5) from the above list of assumptions at step j. The next claim shows that such $\tilde{p}$ can be found.

Claim 6.3.7. *There is a condition $\tilde{p}$ in $\mathbb{P}_{\delta_j} \cap M^{\delta_j}_{\alpha_j}$ obtained from $[p \upharpoonright \delta_j]_{M^{\delta_j}_{\alpha_j}}$ by extending only the collapse part above each α_i, $i < j$, such that*

$$\tilde{p} \Vdash f^{q_i}_{\delta_i}(\bar{s}) <_{\dot{T}} f^p_{\delta_i}(s),$$

for any $i < j$ and $s \in A_i$ such that $f^p_{\delta_i}(s) \in V_{\alpha_j}$. Moreover, we may assume that $\tilde{p}$ is still super-nice with respect to each $M^{\delta_i}_{\alpha_i}$, $k \leqslant i < j$.

Proof of Claim 6.3.7. By Lemma 6.2.16. ☐

Let $\tilde{p}$ be as in Claim 6.3.7.

We continue the construction and consider separately the cases $j = \iota$ and $j < \iota$.

If $j = \iota$, then $\delta_\iota = \delta$ and $\alpha_\iota = \kappa$. In this case let q_ι be the pointwise union of the q_i, $i < \iota$. Let p_ι be the condition obtained by taking the pointwise union of the conditions p_i, $i < j$, and $\tilde{p}$, and furthermore, whenever $\xi \in \gamma \cap M^\delta_\alpha$, then extend the

ξ-th coordinate by

$$f_\xi := (f_\xi^{q_i} \cap V_\alpha) \cup \bigcup_{i<j, \xi \in \mathsf{sp}(p_i)} f_\xi^{p_i} \cup (f_\xi^p \cap V_{\alpha_j}).$$

It follows by construction, as in the proof of strong properness (Lemma 6.2.18), that p_ι is a condition in $\mathbb{P}_{\delta_j} \cap M_{\alpha_j}^{\delta_j}$, and it follows by construction that p_ι and q_ι nicely split and $[p_\iota]_{M_\alpha^\delta} = [q_\iota]_{M_\alpha^\delta}$.

Suppose then that $j < \iota$. We construct p_j and q_j in multiple steps.

Claim 6.3.8. *There are conditions $p_j^0 \in \mathbb{P}_{\delta_j} \cap M_{\alpha_j}^{\delta_j}$ and $q_j^0 \in \mathbb{P}_\gamma$ such that:*

1. *$p_j^0 \leqslant p_i \restriction \delta_j$ for every $i < j$ and $p_j^0 \leqslant \tilde{p}$,*
2. *p_j^0 and $q_j^0 \restriction \delta_j$ nicely split and super-nice with respect to M_α^δ and $[p_j^0]_{M_\alpha^\delta} = [q_j^0 \restriction \delta_j]_{M_\alpha^\delta}$,*
3. *p_j^0 decides meets and their implicit images and relevant exit nodes in the set $(\mathsf{dom}(f_{\delta_j}^p) \cap M_{\alpha_j}^{\delta_j}) \cup \bigcup_{i<j, \delta_j \in \mathsf{sp}(p_i)} \mathsf{dom}(f_{\delta_j}^{p_i})$, as well as exit nodes below nodes $s \in \mathsf{dom}(f_{\delta_j}^p) \cap M_{\alpha_j}^{\delta_j}$ from structures that appear in p_i at coordinate δ_j, for $i < j$ such that $\delta_j \in \mathsf{sp}(p_i)$, as in item (5) of the definition of multi-extension 6.2.14,*
4. *q_j^0 extends every q_i, $i < j$, and agrees with their pointwise union after δ_j, i.e. $q_j^0 \restriction [\delta_j, \delta) = \bigcup_{i<j} q_i \restriction [\delta_j, \delta)$.*

Proof of Claim 6.3.8. Let $G \subseteq \mathbb{P}_{\delta_j} \cap M_\alpha^\delta$ be a generic filter that contains the common trace $[p_i \restriction \delta_j]_{M_\alpha^\delta} = [q_i \restriction \delta_j]_{M_\alpha^\delta}$ for every $i < j$. By assumption every p_i and q_i is super-nice with respect to M_α^δ, and thus $p_i, q_i \restriction \delta_j \in \mathbb{P}_{\delta_j}/G$ for every $i < j$. Also $\tilde{p} \in \mathbb{P}_{\delta_j}/G$, as every $[p_i \restriction \delta_j]_{M_\alpha^\delta}$, $i < j$, is a residue of $\tilde{p}$ into M_α^δ. By strong properness for δ_j-sequences in the quotient, Lemma 6.2.25, there is a condition $p_j^{0,0} \in (\mathbb{P}_{\delta_j} \cap M_{\alpha_j}^{\delta_j})/G$ that extends $\tilde{p}$ and each $p_i \restriction \delta_j$, $i < j$. Up to extending $p_j^{0,0}$ in the quotient $\mathbb{P}_{\delta_j}/G$ we may assume that it decides the relevant meets and exit nodes from item (3). Let q_j^{00} be the simple amalgamation of the $q_i \restriction \delta_j$, $i < j$. Then also $q_j^{00} \in \mathbb{P}_{\delta_j}/G$. Furthermore, up to extending p_j^{00} and q_j^{00} in the quotient $\mathbb{P}_{\delta_j}/G$, we may assume that they are super-nice with respect to M_α^δ and have the same trace to it, $[p_j^{00}]_{M_\alpha^\delta} = [q_j^{00} \restriction \delta_j]_{M_\alpha^\delta}$.

Now, p_j^{00} and q_j^{00} are super-nice with respect to M_α^δ and have same trace to it. Since $j < \iota$ by assumption, we have $\delta_j < \gamma$, as argued in the beginning of the proof. By induction hypothesis for δ_j applied to the conditions p_j^{00} and $q_j^{00} \restriction \delta_j$, we find

$p_j^{01} \leqslant p_j^{00}$ and $q_j^{01} \leqslant q_j$ that nicely split with respect to M_α^δ and have same trace to it. By the 'Moreover' -part of induction hypothesis, we may assume that $p_j^{01} \in M_{\alpha_j}^{\delta_j}$.

By the first item of the lemma, there are $p_j^{02} \leqslant p_j^{01}$ and $q_j^{02} \leqslant q_j^{01}$ that are super-nice with respect to M_α^δ and have same trace to it, and by the 'Moreover' -part, we may assume also that $p_j^{02} \in M_{\alpha_j}^{\delta_j}$. Continuing ω many times, we find conditions p_j^{0n} and q_j^{0n} such that $[p_j^{0n}]_{M_\alpha^\delta} = [q_j^{0n}]_{M_\alpha^\delta}$, and for even $n < \omega$, p_j^{0n} and q_j^{0n} are super-nice with respect to M_α^δ and for odd $n < \omega$, they nicely split with respect to M_α^δ.

Let p_j^0 be the simple amalgamation of the p_j^{0n}, $n < \omega$, and let q_j^0 be the simple amalgamation of the q_j^{0n}, $n < \omega$, together with the q_i, $i < j$. Then p_j^0 and q_j^0 are as wanted. In particular, $p_j^0 \in M_{\alpha_j}^{\delta_j}$.

□

Let p_j^0 and q_j^0 be as in Claim 6.3.8. By assumption p_j^0 and $q_j^0 \restriction \delta_j$ nicely split and are super-nice with respect to M_α^δ and satisfy $[p_j^0]_{M_\alpha^\delta} = [q_j^0 \restriction \delta_j]_{M_\alpha^\delta}$, and furthermore, the sequence

$$(p_i \restriction \min\{\delta_i, \delta_{j+1}\} : i < j)^\frown(p_j^0 \restriction \min\{\delta_j, \delta_{j+1}\})$$

forms a $[p \restriction \delta_{j+1}]_{M_{\alpha_{j+1}}^{\delta_{j+1}}}$-multi-extension.

In case $\delta_j \notin \gamma \cap M_\alpha^\delta$, we end step j of the recursion here and let $p_j := p_j^0$ and $q_j := q_j^0$. Otherwise, we assume from now onwards that $\delta_j \in \gamma \cap M_\alpha^\delta$. We make few further extensions to p_j^0 and q_j^0.

Claim 6.3.9. *If $t \in V_{\alpha_j}$, then there are extensions $p_j^1 \leqslant p_j^0$ and $q_j^1 \leqslant q_j^0$ that split t, and satisfy that p_j^1 and q_j^1 nicely split and are super-nice with respect to M_α^δ, have same trace to it, and $p_j^1 \in M_{\alpha_j}^{\delta_j}$.*

Proof of Claim 6.3.9. Since p_j^0 and q_j^0 are super-nice with respect to M_α^δ and have same trace to it, we may fix a generic $G \subseteq \mathbb{P}_{\delta_j} \cap M_\alpha^\delta$ with $p_j^0, q_j^p \restriction \delta_j \in \mathbb{P}_{\delta_j}/G$. By assumption $\delta_j < \delta$, so by induction hypothesis (third item of the lemma) the tree $\dot{T}$ is a wide κ-Aronszajn tree in $V^{\mathbb{P}_{\delta_j}}$. As in Lemma 6.3.2, we argue that the branch below t cannot be introduced by $\mathbb{P}_{\delta_j} \cap M_\alpha^\delta$: it follows from the case assumption $\delta_j \in \gamma \cap M_\alpha^\delta$ that $\mathbb{P}_{\delta_j} \in M_\alpha^\delta$, so in particular M_α^δ reflects the statement "$\Vdash_{\mathbb{P}_{\delta_j}} \dot{T}$ is wide κ-Aronszajn tree". Hence the branch below t cannot be introduced by G. Thus there are two extensions $p^L, p^R \leqslant p_j^0$ in $\mathbb{P}_{\delta_j}/G$ that split t at some level $\bar{\alpha} < \alpha$ with some nodes $t^L \neq t^R$. By elementarity we may assume that $p^L, p^R \in M_{\alpha_j}^{\delta_j}$. Extend $q' \leqslant q_j^0 \restriction \delta_j$ in $\mathbb{P}_{\delta_j}/G$ to decide the predecessor of t at level $\bar{\alpha}$, call it $\bar{t}$. If $\bar{t} \neq t^L$, then we let $p' := p^L$ and otherwise we let $p' := p^R$. Then p' and q' split t, extend p^0_j and $q^0_j \restriction \delta_j$, respectively, and are in the quotient $\mathbb{P}_{\delta_j}/G$. Since p' and q' are in the quotient, they can be assumed to be super-nice with respect to M^δ_α and have the same trace to it.

By dovetailing ω many times, alternating the induction hypothesis for δ_j and first item of the lemma, as in the proof of the previous claim, we find $p^1_j \leqslant p'$ and $q^1_j \leqslant q', q^0_j$ that nicely split and are super-nice with respect to M^δ_α and have same trace to it, and moreover, $p^1_j \in M^{\delta_j}_{\alpha_j}$. □

Let p^1_j and q^1_j be as in Claim 6.3.9. They split t in case $t \in V_{\alpha_j}$. We make two more last extensions. Denote

$$\begin{aligned} A_j &= \{s \in \mathsf{dom}(f^p_{\delta_j}) : s \text{ is an exit node from } V_\alpha\} \cap (V_{\alpha_j} - \bigcup_{i<j} V_{\alpha_i}), \\ B_j &= \{s \in \mathsf{dom}(f^{q_j}_{\delta_j}) : s \text{ is an exit node from } V_\alpha\}. \end{aligned}$$

By iterating Lemma 6.3.2, first item of this lemma, and induction hypothesis at δ_j countably many times, we find two conditions $p^2_j \leqslant p^1_j$ and $q^2_j \leqslant q^1_j$ such that

- p^2_j and $q^2_j \restriction \delta_j$ nicely split and are super-nice with respect to M^δ_α and $[p^2_j]_{M^\delta_\alpha} = [q^2_j]_{M^\delta_\alpha}$,
- $p^2_j \in M^{\delta_j}_{\alpha_j}$,
- p^2_j and $q^2_j \restriction \delta_j$ split every pair $(s, s') \in A_j \times B_j$ with some pair $(\bar{s}, \bar{s}')$ of distinct nodes of same height.

For every $i < j$ such that $\delta_j \in \mathsf{sp}(p_i)$, consider the function $f^{p_i}_{\delta_j}$. We have

$$f^{p_i}_{\delta_j} \cap V_\alpha = f^{q_i}_{\delta_j} \cap V_\alpha,$$

by the assumption that $[p_i]_{M^\delta_\alpha} = [q_i \restriction \delta_i]_{M^\delta_\alpha}$ and $\delta_j \in M^\delta_\alpha$. Let

$$f := \bigcup_{i<j, \delta_j \in \mathsf{sp}(p_i)} f^{p_i}_{\delta_j} \cap V_\alpha.$$

Then $f = f^{q^2_j}_{\delta_j} \cap V_\alpha$, by the assumption that $q^2_j \restriction [\delta_j, \delta) = \bigcup_{i<j} q_i \restriction [\delta_j, \delta)$. And f is a level- and meet-preserving tree-embedding

$$f : (\dot{S}_{\delta_j} \cap V_\alpha)^G \to (\dot{T} \cap V_\alpha)^G,$$

for any generic $G \subseteq \mathbb{P}_{\delta_j} \cap M^\delta_\alpha$ that contains the common trace $[p^2_j]_{M^\delta_\alpha} = [q^2_j \restriction \delta_j]_{M^\delta_\alpha}$. Fix one such generic. Note that $\delta_j + 1 \leqslant \delta$ and $\delta_j + 1 \in M^\delta_\alpha$. By Node density Lemma applied in the poset $(\mathbb{P}_{\delta_j+1} \cap M^\delta_\alpha)/G$, there is a function $f' \supseteq f$ whose domain is the closure under meets of the set

$$\mathsf{dom}(f) \cup \{\bar{s}, \bar{s}' : (s, s') \in A_j \times B_j\},$$

and which is level- and meet-preserving tree-embedding from $(\dot{S}_{\delta_j} \cap V_\alpha)^G$ to $(\dot{T} \cap V_\alpha)^G$.

Note also that if $(s, s') \in A_j \times B_j$, then , the meet $\bar{s} \wedge \bar{s}'$ is decided by the generic G, since $\bar{s}, \bar{s}' \in (\dot{S}_{\delta_j} \cap V_\alpha)^G$, and furthermore

$$p^2_j \Vdash s \wedge \bar{s}' = \bar{s} \wedge \bar{s}',$$
$$q^2_j \restriction \delta_j \Vdash \bar{s} \wedge s' = \bar{s} \wedge \bar{s}'.$$

This implies in particular, that if $s \neq s'$, then p^2_j forces that $\bar{s}'$ is not below s. And further, that p^2_j decides the meets in the set

$$\mathsf{dom}(f') \cup (\mathsf{dom}(f^p_{\delta_j}) \cap V_{\alpha_j}) \cup \bigcup_{i<j, \delta_j \in \mathsf{sp}(p_i)} f^{p_i}_{\delta_j}.$$

Recall that if $s \in A_j$, then $f^p_{\delta_j}(s) \notin M^{\delta_j}_{\alpha_j}$, so in particular we will be free to extend the collapse coordinate of p to force $f'(\bar{s} \wedge \bar{s}') < f^p_{\delta_j}(s)$ and $f'(\bar{s}') \not< f^p_{\delta_j}(s)$ in a later step $j' > j$ of the recursion whenever $f^p_{\delta_j}(s) \in V_{\alpha_{j'}}$, exactly as in the first claim 6.3.7 of this proof.

We find final extensions $p^3_j \leqslant p^2_j$ and $q^3_j \leqslant q^2_j \restriction \delta_j$, again dovetailing, such that

- p^3_j and q^3_j nicely split and are super-nice with respect to M^δ_α and $[p^3_j]_{M^\delta_\alpha} = [q^3_j]_{M^\delta_\alpha}$,
- the common trace $[p_j]_{M^\delta_\alpha} = [q_j \restriction \delta_j]_{M^\delta_\alpha}$ decides meets in $\mathsf{dom}(f')$ and forces that f' is level- and meet-preserving tree-embedding from $\dot{S}_{\delta_j}$ to $\dot{T}$.

Then, we let $p_j := p^3_j$ and define the condition q_j by letting $q_j \restriction \delta_j := q^3_j$, $q_j \restriction [\delta_j + 1, \delta) = q^2_j \restriction [\delta_j + 1, \delta)$ and defining the δ_j-th coordinate by letting

$$f^{q_j}_{\delta_j} = f' \cup \bigcup_{i<j} f^{q_i}_{\delta_j}$$

and $N^{q_j}_{\delta_j} = N^{q^2_j}_{\delta_j}$. This ends the j-th step of the recursion.

Letting $\hat{p} := p_\nu$ and $\hat{q} := q_\nu$ ends the proof of item (2): the conditions $\hat{p}$ and $\hat{q}$ split t, they nicely split with respect to M^δ_α, and $[\hat{p}]_{M^\delta_\alpha} = [\hat{q}]_{M^\delta_\alpha}$. We argued in the beginning of the lemma that the 'Moreover' -part holds because the p-closure of M^δ_α and therefore p_ν too must be elements in $M^\gamma_{\hat{\beta}}$.

Proof of item 3.

Assume towards contradiction that $\mathbb{P}_\gamma$ introduces a cofinal branch to $\dot{T}$. Fix a $\mathbb{P}_\gamma$-name $\dot{b}$ and a condition p such that $p \Vdash \dot{b}$ is a cofinal branch in $\dot{T}$. Find α such that $p, \dot{b} \in M^\gamma_\alpha$ and $q \leqslant p$ which is super-nice with respect to M^δ_α. Up to extending q, assume that it forces $\dot{b}(\alpha) = t$ for some node $t \in T$. By item 2 applied to the pair (q, q), there are $\hat{q}_1 \leqslant q$ and $\hat{q}_2 \leqslant q$ that are nicely split with respect to M^γ_α, have the same trace $[\hat{q}_1]_{M^\gamma_\alpha} = [\hat{q}_2]_{M^\gamma_\alpha}$, and split t with some distinct nodes t^L and t^R at some level $\bar{\alpha} < \alpha$. By item 1 and Remark 6.3.6, we may assume that $\hat{q}_1$ and $\hat{q}_2$ are super-nice with respect to M^γ_α and have the same trace to it. The the common trace $r := [\hat{q}_1]_{M^\delta_\alpha} = [\hat{q}_2]_{M^\delta_\alpha}$ is a common residue for $\hat{q}_1$ and $\hat{q}_2$. Let $w \in \mathbb{P}_\gamma \cap M^\delta_\alpha$ extend the common trace r to decide the $\bar{\alpha}$-th node on the branch, say it forces $\dot{b}(\bar{\alpha}) = \bar{t}$. If $\bar{t} \neq t^L$, then w cannot be compatible with $\hat{q}_1$ and if $\bar{t} \neq t^R$, then w cannot be compatible with $\hat{q}_2$. This is in contradiction with the fact that the common trace r was a common residue for $\hat{q}_1$ and $\hat{q}_2$. This ends the proof of the lemma.

□

Theorem 6.3.10. *Suppose that the bookkeeping function is such that for each $\delta < \kappa^+$, the name $\dot{S}_\delta$ is a $\mathbb{P}_\delta$-name for a wide κ-Aronszajn tree. Then the wide tree $\dot{T}$ is a wide κ-Aronszajn tree in $V^{\mathbb{P}_{\kappa^+}}$.*

Proof. By Lemma 6.3.5 item (3) and κ^+-cc of $\mathbb{P}_{\kappa^+}$. □

By κ^+-cc of $\mathbb{P}_{\kappa^+}$, it is possible to choose a bookkeeping function such that for every $\mathbb{P}_{\kappa^+}$-name for a wide κ-Aronszajn tree there is δ such that $\dot{S}_\delta$ is a $\mathbb{P}_\delta$-name for an isomorphic tree. We have thus proved:

Theorem 6.3.11. *Assume that there is a weakly compact cardinal. Then the existence of a universal wide $\aleph_2$-Aronszajn tree is consistent.*

Chapter 7

On linear orders

In this chapter we solve the universality problem in the class

$$\mathcal{L}_\kappa := \text{the class of linear orders of size } \kappa \text{ that do not order-embed } \kappa \text{ nor } \kappa^{-1},$$

where κ is either $\aleph_1$ or an uncountable double successor cardinal, and embeddings are order-preserving maps. The result is obtained by leveraging on results of the previous chapters and exploiting a duality between $\mathcal{L}_\kappa$ and the class of "locally ordered wide κ-Aronszajn trees". Namely, it is possible to convert linear orders in $\mathcal{L}_\kappa$ into wide κ-Aronszajn trees and vice versa (see Lemmas 7.1.9 and 7.1.10). Thus, we call members of $\mathcal{L}_\kappa$ **wide κ-Aronszajn lines**. Unfortunately this conversion is not functorial and indeed, it does not hold directly that universal wide κ-Aronszajn tree converts into a universal wide κ-Aronszajn line. The situation is more delicate, and we have to reproduce the proofs with relevant modifications. In fact, this is not surprising, since already by results of Todorčević [26] and Moore [19], under PFA, there does not exist a universal Aronszajn tree, but a universal Aronszajn line does exist, even if an analogous (non-functorial) duality exists between (narrow) Aronszajn trees and lines. See Section 7.1.2 below for a description of the strategy used here.

The theorems of this chapter result from a collaboration with Haytham Hammud. We thank Professor Omer Ben-Neria for suggestions and discussions.

7.1 Linear orders and trees

Definition 7.1.1.

1. For a tree $T = (T, <_T)$ and a node $t \in T$, we denote
$$\boldsymbol{S(t)} := \text{the set of immediate successors of } t.$$
2. A **locally ordered tree** is a triple
$$T = (T, <_T, <_t)_{t \in T}$$
such that $(T, <_T)$ is a tree and $<_t$ linearly orders the set $S(t)$ for each t. The order $<_t$ is called the **local order at $\boldsymbol{t}$**.

We introduce morphisms between locally ordered trees.

Definition 7.1.2. Let $S = (S, <_S, <_s)_{s \in S}$ and $T = (T, <_T, <_t)_{t \in T}$ be locally ordered trees.

1. A function $f : S \to T$ is a **strong tree-embedding** if it is injective, level-preserving and
$$s <_S t \iff f(s) <_T f(t)$$
for all nodes $s, t \in S$.
2. A function $f : S \to T$ **preserves local orders** if for every $s \in S$ and for all $t, t' \in S(s)$:
$$t <_s t' \iff f(t) <_{f(s)} f(t').$$

We will be interested in the category of locally ordered trees equipped with local order -preserving strong embeddings.

7.1.1 Converting trees to lines and vice versa

It is possible to turn trees into linear orders and vice versa. A locally ordered tree can be turned into a linear order in a canonical way.

Notation 7.1.3. If T is a locally ordered tree and s and t are two incomparable nodes, then the local order $<_{s \wedge t}$ can be lifted canonically to s and t by letting
$$s <_{s \wedge t} t :\iff \bar{s} <_{s \wedge t} \bar{t},$$
where $\bar{s}$ and $\bar{t}$ are, respectively, the unique predecessors of s and t in $S(s \wedge t)$.

Definition 7.1.4. Let T be a locally ordered tree. The linear order $<_{L(T)}$ on T is defined by:

$$s <_{L(T)} t :\iff s <_T t \text{ or } s <_{s \wedge t} t.$$

where $<_{s \wedge t}$ is the local order on $S(s \wedge t)$ lifted to s and t. The **linearization $L(T)$** of T is the linear order $(T, <_{L(T)})$.

The linearization is a functor from the category of locally ordered trees equipped with strong local order -preserving maps to the category of linear orders.

Lemma 7.1.5. *Let S and T be locally ordered trees. Every strong local order -preserving tree-embedding from S to T is an order-embedding from $L(S)$ to $L(T)$.*

Proof. Let $f : S \to T$ be a strong local order -preserving tree-embedding. Let $s, t \in S$. We show that

$$s <_{L(S)} t \iff f(s) <_{L(T)} f(t).$$

Indeed:

- If $s <_S t$, then $f(s) <_T f(t)$, so $f(s) <_{L(T)} f(t)$.
- If $s <_{s \wedge t} t$, then $f(s) <_{f(s \wedge t)} f(t)$, so $f(s) <_{L(T)} f(t)$.
- If $f(s) <_{L(T)} f(t)$, then either $f(s) <_T f(t)$, in which case $s <_S t$, or $f(s) <_{f(s) \wedge f(t)} f(t)$ which means that $f(s) <_{f(s \wedge t)} f(t)$, and implies $s <_{s \wedge t} t$.

This is enough for obtaining the desired conclusion. ☐

We saw that a locally ordered tree can be seen as a linear order. Conversely, a linear order can also be turned into a tree. This operation, contrary to the linearization, is not functorial.

Definition 7.1.6. Let L be a linear order. A **partition tree** of L is a tree $T = (T, <_T)$ such that

1. the domain of the tree T is a collection of non-empty intervals of L and each level consists of pairwise disjoint intervals,
2. the tree-order $<_T$ is inverse inclusion,
3. any interval I in T of size at least two has at least two immediate successors,

4. for every point $a \in L$ the singleton $\{a\}$ belogns to T.

In a partition tree T, each successor set $S(t)$ can be ordered via

$$I <_t J \quad :\Longleftrightarrow \qquad \text{every point in the interval } I \text{ is } <_L \text{-below every point in } J.$$

These orders on successor sets of a fixed partition tree is referred to as the **canonical local orders** of the partition tree.

Lemma 7.1.7. *Let L be a linear order and let T be its partition tree locally ordered with the canonical local orders. The map*

$$L \to L(T), \quad a \mapsto \{a\}$$

is an injective order-embedding.

Every linear order L has a (not necessarily unique) partition tree. A partition tree of a linear order L has size at least $|L|$ and most $2^{<|L|}$. They can be constructed as follows. Let L be a linear order and denote $\kappa := |L|$. By recursion on $\alpha < \kappa$, we choose an interval $t_s \subseteq L$ for each $s \in 2^{<\kappa}$:

- Step 0: Let $t_\varnothing := L$,
- Successor step $\alpha + 1$: Let $s \in 2^\alpha$ and assume that t_s is a non-empty interval of L with endpoints a_s and b_s (the case $a_s = \infty$ or $b_s = \infty$ is allowed). Choose a point c in the interval (a_s, b_s) and define

  $$\begin{aligned} t_{s^\frown 0} &:= (a_s, c], \\ t_{s^\frown 1} &:= (c, b_s]. \end{aligned}$$

  (If $b_s = \infty$, then $t_{s^\frown 1} = (c, \infty)$.)
- Limit step β: Let $s \in 2^\beta$ and define

  $$t_s := \bigcap_{\alpha<\beta} t_{s\restriction\alpha}.$$

Finally, let

$$T := \{t_s : s \in 2^{<|L|}, t_s \neq \varnothing\}.$$

Then T ordered by

$$s <_T t :\Longleftrightarrow \ s \supsetneq t.$$

is a binary partition tree of L.

We now connect wide κ-Aronszajn trees with a certain class of linear orders.

Definition 7.1.8. A **wide κ-Aronszajn line** is a linear order L of size $\leqslant \kappa$ such that

1. $\kappa = (\kappa, \in)$ does not order-embed into L,
2. $\kappa^{\text{op}} = (\kappa, \in^{-1})$ does not order-embed into L.

We seek for a **universal** wide κ-Aronszajn line, i.e. a wide κ-Aronszajn line that contains an isomorphic copy of every other wide κ-Aronszajn line.

Lemma 7.1.9. *A partition tree of a wide κ-Aronszajn line is a wide κ-Aronzajn tree.*

Proof. Let T be a partition tree of a wide κ-Aronszajn line L. The fact that T does not have a branch of length κ follows from the fact that such a branch would induce an embedding of κ or κ^{-1} into L. ☐

A wide κ-Aronszajn tree T has **wide κ-Aronszajn local orders** if each local order $(S(t), <_t)$ is a wide κ-Aronszajn line, for any $t \in T$. Binary trees trivially have wide κ-Aronszajn local orders as $|S(t)| = 2$ for every node t.

The name of wide κ-Aronszajn lines is justified by the following lemma, that together with Lemma 7.1.9 shows that in some sense, wide Aronszajn trees and lines correspond to each other.

Lemma 7.1.10. *Let κ be a regular cardinal. If T is a wide κ-Aronszajn tree with wide κ-Aronszajn local orders, then $L(T)$ is a wide κ-Aronszajn line.*

Proof. Let T be a wide κ-Aronszajn tree with wide κ-Aronszajn local orders. Suppose towards contradiction that there is an order-preserving embedding

$$f : (\kappa, \in) \to (T, <_{L(T)}).$$

Let

$$B := \{t \in T : \text{there are } \kappa \text{ many nodes } s \in \mathsf{ran}(f) \text{ such that } t \leqslant_T s\}.$$

We show first that B is linearly ordered by the tree order $<_T$. To this end, let $t, s \in B$. Suppose that they are incompatible in $<_T$. Denote by u their meet $t \wedge s$. Suppose by symmetry that $t <_u s$. But now, every node above t is $<_{L(T)}$-below s. In particular, since t is below κ many nodes in $\mathsf{ran}(f)$, there are κ many nodes in $\mathsf{ran}(f)$ below s. This means that any node in $\mathsf{ran}(f)$ above s has κ many $<_{L(T)}$-predecessors in $\mathsf{ran}(f)$, which contradicts the assumption that $(\mathsf{ran}(f), <_{L(T)})$ has order-type $(\kappa, \in)$.

Hence B is linearly ordered, in other words it is a branch.

We note first that B cannot have a maximal element. To this end, suppose that $t \in B$ is a maximal element of B. Now there are κ many nodes $s \in \mathsf{ran}(f)$ such that $t <_T s$. By assumption κ does not order-embed into the successor set $S(t)$, which implies that there are only $< \kappa$ many nodes in $S(t)$ that are below some node in $\mathsf{ran}(f)$. It follows by pigeonhole principle that there is a node in $S(t)$ which has κ many nodes $<_T$-above it in $\mathsf{ran}(f)$, which contradicts the maximality of t. Hence B does not have a maximal element.

We show that the order-type of B is κ. For each node $t \in B$, let W_t be the set of nodes $s \in \mathsf{ran}(f)$ that are $<_T$-above t. Each W_t is a subset of $\mathsf{ran}(f)$ of size κ. For nodes $s, t \in B$ that satisfy $s <_T t$, we have the following:

1. There are no nodes $t' \in \mathsf{ran}(f)$ such that $t \wedge t' = s$ and $t <_{L(T)} t'$. This is because there are κ many nodes $<_T$-above t in $\mathsf{ran}(f)$ that all would be $<_{L(T)}$-below t'.
2. There are strictly less than κ many nodes $t' \in \mathsf{ran}(f)$ such that $s <_T t' <_{L(T)} t$.

It follows from these that $|W_s - W_t| < \kappa$ for nodes $s <_T t$ from B. However, we also have the equality

$$\bigcup_{t \in B} W_t = \bigcup \{W_s - W_t : s, t \in B, s <_T t\}$$

The left-hand-side has size κ, and the right-hand-side is a union of length $|B|$ of sets of size $< \kappa$. By regularity of κ, we must have $|B| = \kappa$. We have found a branch of order-type κ in T, a contradiction.

□

We have shown that whenever κ is a regular cardinal, wide κ-Aronszajn trees correspond to wide κ-Aronszajn trees in the following sense:

- If T is a wide κ-Aronszajn tree, we can assign a wide κ-Aronszajn local order to each successor set $S(t)$, and obtain a wide κ-Aronszajn line $L(T)$ by linearizing it, as in Lemma 7.1.10.
- If L is a wide κ-Aronszajn line, then each of its partition trees is a wide κ-Aronszajn tree with wide κ-Aronszajn local orders.

Furthremore, any wide κ-Aronszajn line L order-embeds into the linearization $L(T)$ of any of its partition trees T.

7.1.2 Strategy

Our goal is to show that, consistently, there is a universal wide μ^+-Aronszajn line for any $\mu \in \{\aleph_0\} \cup \{\lambda^+ : \lambda \in \mathsf{Card}\}$. We argue here that it is enough to show that, consistently, there is a wide μ^+-Aronszajn tree with wide μ^+-Aronszajn local orders which is universal for the class of wide binary μ^+-Aronszajn locally ordered trees. We simplify the notation:

Notation 7.1.11. Let κ be a regular cardinal.

1. $\boldsymbol{\mathcal{K}_\kappa}$ is the category where objects are wide κ-Aronszajn trees with wide κ-Aronszajn local orders and morphisms are strong local-order -preserving tree-embeddings.
2. $\boldsymbol{\mathcal{K}^2_\kappa}$ is the subcategory of $\mathcal{K}_\kappa$ consisting of those trees in $\mathcal{K}_\kappa$ that are binary.
3. A tree $T^* \in \mathcal{K}_\kappa$ is **universal for $\boldsymbol{\mathcal{K}^2_\kappa}$**, if every tree in $\mathcal{K}^2_\kappa$ embeds into T^* (via a strong local-order -preserving tree-embedding).

Remark 7.1.12. Recall the class $\mathcal{T}_\kappa$ of wide κ-Aronszajn trees equipped with strong tree-embeddings. Clearly every member of $\mathcal{K}_\kappa$ can be functorially turned into a member of $\mathcal{T}_\kappa$, simply by forgetting the local orders. Conversely, every member of $\mathcal{T}_\kappa$ can be turned into a member of $\mathcal{K}_\kappa$, by selecting a local order to each successor set (not functorially).

Phrased using the new notation, we now argue that in order to find a universal wide κ-Aronszajn line - i.e. a wide κ-Aronszajn line into which every other such order-embeds - it suffices to find a tree $T^* \in \mathcal{K}_\kappa$ which is universal for the class $\mathcal{K}^2_\kappa$ under strong embeddings.

Lemma 7.1.13. *If there is a tree $T \in \mathcal{K}_\kappa$ that is universal for $\mathcal{K}^2_\kappa$, then there is a universal wide κ-Aronszajn line.*

Proof. Suppose that $T^* \in \mathcal{K}_\kappa$ is universal for $\mathcal{K}^2_\kappa$. We claim that $L(T^*)$ is a universal wide κ-Aronszajn line. Let L be a wide κ-Aronszajn line. By Lemmas 7.1.9 and 7.1.7 there is a binary wide κ-Aronszajn tree T and an embedding

$$L \hookrightarrow L(T).$$

Since T^* is universal for $\mathcal{K}^2_\kappa$, there is an embedding

$$T \hookrightarrow T^*.$$

By Lemma 7.1.5, such an embedding witnesses

$$L(T) \hookrightarrow L(T^*).$$

By composing, we find an embedding $L \hookrightarrow L(T^*)$. This gives the desired conclusion: $L(T^*)$ is a universal κ-Aronszajn line. $\square$

The rest of the paper is devoted for proving that assuming the existence of a weakly compact cardinal κ, it is consistent that for any $\mu \in \{\aleph_0\} \cup \{\lambda^+ : \lambda \in \mathsf{Card} \cap \kappa\}$ relative to a the there is a tree $T^* \in \mathcal{K}_{\mu^+}$ that is universal for $\mathcal{K}^2_{\mu^+}$.

7.2 Poset

As said in Lemma 7.1.13, following Notation 7.1.11, in order to obtain a universal object in $\mathcal{L}_\kappa$, it suffices to force an object into $\mathcal{K}_{\mu^+}$ that is universal for objects in $\mathcal{K}^2_{\mu^+}$. The rest of the chapter is devoted to do this. The forcing construction is analogous to the one in chapters 5 and 6: we begin with a weakly compact cardinal κ, produce a "flexible" member $\dot{T}$ of $\mathcal{K}_{\mu^+}$ in a $\mathsf{Col}(\mu, <\kappa)$-extension, and then iterate further to create embeddings into $\dot{T}$ from all members of $\mathcal{K}^2_{\mu^+}$. It is essential that the bookkeeping function chooses only members in $\mathcal{K}^2_{\mu^+}$, i.e. trees that are binary. Otherwise our arguments do not work as such. See the beginning of Section 7.3 for discussion on the matter.

We begin with the definition of the "flexible locally ordered tree $\dot{T}$".

Assumption 7.2.1. *From now onwards, we assume that* GCH *holds in V For the sake of notational simplicity* ***μ is either $\aleph_0$ or an uncountable successor cardinal, and κ is a weakly compact cardinal above μ****. Moreover, we fix a* $\mathsf{Col}(\mu, <\kappa)$*-name* $\dot{\boldsymbol{\mu}}$ *for a wide κ-Aronszajn tree and width and label maps $t \mapsto$* $\mathsf{wd}(t)$ *and $t \mapsto \ell(t)$, as in Proposition 4.2.1.*

Notation 7.2.2. Let $\dot{\boldsymbol{L}}$ be a $\mathsf{Col}(\mu, <\kappa)$-name for the set of all non-constant functions $\mu \to 2$ in $V^{\mathsf{Col}(\mu,<\kappa)}$, ordered linearly by $a <_{\dot{L}} b$ if $a(\alpha) < b(\alpha)$ where α is the least ordinal where a and b disagree.

Remark 7.2.3. In $V^{\mathsf{Col}(\mu,<\kappa)}$, the linear order $\dot{L}$ has size $2^\mu = \mu^+ = \kappa$ and it is a wide κ-Aronszajn line, being the linearization of a wide κ-Aronszajn tree. It is dense and has no endpoints. Furthermore, the underlying wide κ-Aronszajn tree of $\dot{L}$ is *special*, which implies in particular that no forcing that preserves κ can add an order-preserving embedding from $(\kappa, \in)$ or $(\kappa, \in^{-1})$ into $\dot{L}$. In other words, $\dot{L}$ remains a wide κ-Aronszajn line in any further forcing extension $V^{\mathsf{Col}(\mu,<\kappa)*\dot{\mathbb{Q}}}$ that preserves κ.

Definition 7.2.4. In $V^{\mathsf{Col}(\mu,<\kappa)}$, we equip the tree $\dot{T}$ with the following local orders: for each $t \in T$, we let $\dot{<}_t$ be a linear order on the successor set $S(t)$ such that

$$\Vdash_{\mathsf{Col}(\mu,<\kappa)} ``(S(t), \dot{<}_t) \cong \dot{L}",$$

where $\dot{L}$ is the wide κ-Aronszajn line from 7.2.2.

From now onwards, when we refer to $\dot{T}$, we refer to $\dot{T}$ equipped with the local orders from Definition 7.2.4.

We will next define the $(<\mu)$ -support iteration $(\mathbb{Q}_\delta : \delta \leqslant \kappa^+)$ intended to create embeddings into $\dot{T}$.

We will first fix a bookkeeping function. We intentionally leave some flexibility regarding it, for instance, for now we will only assume that it picks locally ordered wide κ-trees that are binary, but not necessarily Aronszajn.

Notation 7.2.5. Fix a bookkeeping function

$$\kappa^+ \to H_{\kappa^+}, \quad \gamma \mapsto \dot{S}_\gamma$$

such that whenever the poset $\mathbb{Q}_\gamma$ has been defined, then $\dot{S}_\gamma$ is a $\mathbb{Q}_\gamma$-name for a normal binary tree with domain $\kappa \times \kappa$ and that $\Vdash_{\mathbb{Q}_\gamma} \mathsf{Lev}_\alpha(\dot{S}_\gamma) = \{\alpha\} \times \kappa$.

When defining the iteration $(\mathbb{Q}_\delta : \delta \leqslant \kappa^+)$, we follow the convention that $\mathbb{Q}_0 = \{\varnothing\}$. The first nontrivial poset $\mathbb{Q}_1$ will (essentially) be $\mathsf{Col}(\mu, <\kappa)$, and the final poset $\mathbb{Q}_{\kappa^+}$ adds a local-order preserving tree-embedding from every binary wide κ-Aronszajn tree into $\dot{T}$. As in chapters 5 and 6, the posets $\mathbb{Q}_\delta$, $\delta \leqslant \kappa^+$, will be defined by recursion on $\delta < \kappa^+$.

Notation 7.2.6. Once a poset $\mathbb{Q}_\delta$ is defined for $\delta < \kappa^+$, we let $\mathcal{E}_\delta$ and $\{M^\delta_\alpha : \alpha \in \mathcal{E}_\delta\}$ be as in Definition 4.1.4.

We assume without loss of generality that each model M^δ_α contains the bookkeeping function, the locally ordered tree $\dot{T}$ as well as the width and label functions $t \mapsto \mathsf{wd}(t)$ and $t \mapsto \ell(t)$.

Before giving the definition of the posets $\mathbb{Q}_\delta$, we need to name two notions related to trees:

Definition 7.2.7. Let S be a tree.

1. The **effective meet** of nodes $s, t \in S$ is the set of size at most three that consists of the meet $s \wedge t$ and the two immediate successors $\bar{s}$ and $\bar{t}$ of the meet $s \wedge t$ that satisfy $\bar{s} \leqslant_S s$ and $\bar{t} \leqslant_S t$. The effective meet of s and t is denoted by $\boldsymbol{m(s,t)}$.

2. A node $s \in S$ is an **exit node** from a set M if $s \notin M$ but the branch below s is contained in M, i.e. $b_s = \{t : t <_S s\} \subseteq S \cap M$.

A set $A \subseteq S$ is **closed under effective meets** if it holds for any nodes s and t, that if $s, t \in A$, then $m(s,t) \subseteq A$.

We are ready to define the posets $\mathbb{Q}_\delta$, $\delta \leqslant \kappa^+$. Recursively, when defining $\mathbb{Q}_\delta$, we assume that the models $(M^\gamma_\alpha : \alpha \in \mathcal{E}_\gamma)$, from Definition 4.1.4 have already been defined for every $\gamma < \delta$. The definition is almost the same as Definition 5.1.4 in case $\mu = \aleph_0$ and as Definition 6.1.3 in case $\mu = \aleph_1$, with the only exception that the tree-embedding approximations are required to respect the local orders. Note that the bookkeeping function picks only trees $\dot{S}_\delta$ that are forced to be binary, which is not (necessarily) the case in chapters 5 and 6. This will be essential in arguments that require amalgamating two conditions while preserving the fact that the embeddings respect local orders.

Definition 7.2.8. Let $\mu \in \{\aleph_0, \aleph_1\}$ and let $\delta < \kappa^+$. Conditions in the poset $\mathbb{Q}^\mu_\delta$ are functions

$$p : \delta \to V_\kappa$$

such that each $p(\gamma)$ is a pair

$$p(\gamma) = (f^p_\gamma, N^p_\gamma),$$

satisfying

1. $f^p_0 \in \mathsf{Col}(\mu, <\kappa)$,
2. $f^p_\gamma : \mathsf{dom}(S_\gamma) \to \mathsf{dom}(T_\gamma)$, for non-zero $\gamma < \delta$, is a partial injective map of size $|f^p_\gamma| < \mu$ such that:
    (a) $p \restriction \gamma$ decides the effective meets and local orders in the set $\mathsf{dom}(f^p_\gamma)$,
    (b) $p \restriction \gamma \Vdash$ ``$\mathsf{dom}(f^p_\gamma) \subseteq \dot{S}_\gamma$ is closed under effective meets",
    (c) $p \restriction \gamma \Vdash$ ``$f^p_\gamma : \dot{S}_\gamma \to \dot{T}_\gamma$ is an injective tree-embedding that preserves levels, effective meets and local orders",
3. $N^p_\gamma \subseteq \lim \mathcal{E}_\gamma$ is a set of size $|N^p_\gamma| < \mu$ such that whenever $\alpha \in N^p_\gamma$ and $\xi \in \gamma \cap M^\gamma_\alpha$, then $\alpha \in N^p_\xi$, and the union

    $$\bigcup_{\gamma<\delta} N^p_\gamma$$

    has size $< \mu$,

4. for every non-zero γ, every $s \in \mathsf{dom}(f^p_\gamma)$ and $\alpha \in N^p_\gamma$:
    (a) $s \in M^\gamma_\alpha$ if and only if $f^p_\gamma(s) \in M^\gamma_\alpha$,
    (b) $p \restriction \gamma \Vdash$ "s is an exit node from M^γ_α if and only if $f^p_\gamma(s)$ is",
    (c) if $s \notin M^\gamma_\alpha$, then there is $\bar{s} \in \mathsf{dom}(f^p_\gamma) - M^\gamma_\alpha$ such that
    $$p \restriction \gamma \Vdash \text{``}\bar{s} \leqslant_{\dot{S}_\gamma} s \text{ and } \bar{s} \text{ is an exit node from } M^\gamma_\alpha\text{''},$$
    (d) if $p \restriction \gamma \Vdash$ "s is an exit node from M^γ_α" and $\mathsf{ht}(s)$ is a successor ordinal, then $p \restriction \gamma$ decides the immediate predecessor $\bar{s}$ of s and $\bar{s} \in \mathsf{dom}(f^p_\gamma)$,
    (e) if $p \restriction \gamma \Vdash$ "s is an exit node from M^γ_α", then $p \restriction \gamma$ decides the ordinal
    $$\mathsf{wd}(s) := \text{the least } \beta \text{ in } \lim \mathcal{E}_\gamma - \mathsf{ht}(s) \text{ such that } b_s \subseteq \mathsf{ht}(s) \times \beta,$$
    and it satisfies $\mathsf{wd}(s) = \mathsf{wd}(f^p_\gamma(s))$.
    (f) if $p \restriction \gamma \Vdash$ "s is an exit node from M^γ_α", then the label $\beta := \ell(f^p_\gamma(s))$ satisfies:
        i. $\beta \in \mathcal{E}_\gamma$ and $\beta \in N^p_\xi$ for every $\xi \in M^\gamma_\beta$,
        ii. $s \in M^\gamma_\beta$.
5. the **support** $\mathsf{sp}(p) := \{\gamma \in \delta : f^p_\gamma \neq \varnothing\}$ has size $< \mu$.

The order is pointwise inverse inclusion: $q \leqslant p$ if $f^q_\gamma \supseteq f^p_\gamma$ and $N^q_\gamma \supseteq N^p_\gamma$ for all $\gamma < \delta$.

Remark 7.2.9. The set $\{\gamma < \delta : N^p_\gamma \neq \varnothing\}$ is not necessarily of size $< \mu$, even if the union $\bigcup_{\gamma<\delta} N^p_\gamma$ is.

Remark 7.2.10. As in the case of the posets to force a universal $\aleph_1$- or $\aleph_2$-Aronszajn tree (Definitions 5.1.4 and 6.1.3), also here it holds that if $f^p_\gamma(s) = t$ and s and t are exit nodes from a model M^γ_α with $\alpha \in N^p_\gamma$, then $\ell(t) \in \mathcal{E}_\gamma$ and the model $M^\gamma_{\ell(t)}$ separates s and t in the sense that
$$s \in M^\gamma_{\ell(t)} \text{ and } t \notin M^\gamma_{\ell(t)}.$$
Moreover, $\ell(t) \in N^p_\xi$ for every $\xi \in \gamma \cap M^\gamma_{\ell(t)}$, but clearly $\ell(t) \notin N^p_\gamma$.

Remark 7.2.11. Each $\mathbb{Q}^\mu_\delta$, $\delta < \kappa^+$, has size κ and can thus be coded as a subset of V_κ using any injection $\delta \to \kappa$. We tacitly identify each $\mathbb{Q}^\mu_\delta$ with an isomorphic poset contained in V_κ, and assume that this identification was done using the $<_\theta$-least injection $\delta \to \kappa$, that belongs to each model M^δ_α, $\alpha < \kappa$.

The posets $(\mathbb{Q}^\mu_\delta : \delta \leqslant \kappa^+)$ have now been defined. Note that $\mathbb{Q}^\mu_0 = \{\varnothing\}$ and $\mathbb{Q}^\mu_1$ contains $\mathrm{Col}(\mu, <\kappa)$ as a complete subposet; if $(f, N) \in \mathbb{Q}^\mu_1$ and $g \in \mathrm{Col}(\mu, <\kappa)$ extends f, then $(g, N) \in \mathbb{Q}^\mu_1$ and $(g, N) \leqslant (f, N)$.

The goal of the rest of the paper is to show that the poset $\mathbb{Q}^\mu_{\kappa^+}$ has the following properties:

1. $\mathbb{Q}^\mu_{\kappa^+}$ is $<\mu$-closed,
2. $\mathbb{Q}^\mu_{\kappa^+}$ has κ^+-cc, so it preserves all cardinals $\lambda \geqslant \kappa^+$,
3. $\mathbb{Q}^\mu_{\kappa^+}$ collapses every cardinal $\alpha \in [\mu, \kappa)$ onto μ,
4. $\mathbb{Q}^\mu_{\kappa^+}$ preserves κ and adds κ^+ many subsets of μ,
5. there is an injective local order and level -preserving tree-embedding $\dot{S}_\gamma \hookrightarrow \dot{T}$ in $V^{\mathbb{Q}^\mu_{\kappa^+}}$, for every $\gamma < \kappa^+$,
6. if the bookkeeping function picks only names for binary wide κ-Aronszajn locally ordered trees, then $\dot{T}$ is a wide μ^+-Aronszajn tree in $V^{\mathbb{Q}^\mu_{\kappa^+}}$.

The first three properties are almost immediate from the definition of $\mathbb{Q}^\mu_{\kappa^+}$. They are proved exactly as in Chapters 5 and 6. The last three are proved in the rest of the sections. Strong properness of each $\mathbb{Q}^\mu_\delta$ with respect to the models $\{M^\delta_\alpha : \alpha \in \mathcal{E}_\delta\}$ will guarantee both the preservation of κ and the fact that the powerset of μ will be pushed up to μ^{++} in $V^{\mathbb{Q}^\mu_{\kappa^+}}$. In particular, with a suitable bookkeeping function, it will hold in $V^{\mathbb{Q}^\mu_{\kappa^+}}$ that $\dot{T}$ is universal for $\mathcal{K}^2_{\mu^+}$, $\mu^+ = \kappa$ and $2^\mu = \mu^{++} = \kappa^+$. In the light of discussion in subsection 7.1.2, the linearization $L(\dot{T})$ will be a universal wide μ^+-Aronszajn line in $V^{\mathbb{Q}^\mu_{\kappa^+}}$.

The proofs follow closely the proofs in Chapter 5 in case $\mu = \omega$ and the proofs in Chapter 6 in case μ is a successor cardinal. It was not possible to directly quote the relevant propositions, since our poset here is slightly more complicated due to the addition of local orders. However, only minor modifications are required in proofs to accommodate our setup. The modifications are as follows: whenever we add a node in the domain of en embedding approximation of a condition, we need to add another node below it in order to take care of the requirement that the domain is closed under effective meets. It is not surprising that this can be done once we have proved that in the case of the posets $\mathbb{Q}^\mu_\delta$ as well, it is easy to add nodes to the domains of embedding functions that are forced to be below nodes already in the domain (see Lemma 7.2.14 below).

We deal with the cases $\mu = \omega$ and μ an uncountable successor cardinal in two separate sections. We could not find a unified framework. We provide proofs for node density and strong properness, but leave the preservation of Aronszajnness for the reader, as the modifications there are minor and very similar to the proofs of node density and strong properness, but the proof is significantly longer.

As in Chapters 5 and 6, we have the sets E^p_δ:

Definition 7.2.12. Let $\delta < \kappa^+$ and $p \in \mathbb{Q}^\mu_\delta$. Let

$$\boldsymbol{E^p_\delta} := \{\alpha \in \mathcal{E}_\delta : \alpha \text{ belongs to } N^p_\gamma \text{ for every } \gamma \in \delta \cap M^\delta_\alpha\}$$

We write E^p_γ for $E^{p\restriction\gamma}_\gamma$.

If $p \in \mathbb{Q}^\mu_\delta$ and $\gamma < \delta$, then $N^p_\gamma \subseteq E^p_\gamma$. This follows from definitions. By assumption, N^p_γ consists of limit points of $\mathcal{E}_\gamma$, but E^p_γ might contain successor points of $\mathcal{E}_\gamma$ as well.

And as in Chapters 5 and 6, in order to show the strong properness of the poset $\mathbb{Q}^\mu_\delta$ will be proved by showing the following two items for every $\delta < \kappa^+$ and $\alpha \in \mathcal{E}_\delta$:

1. whenever $p \in \mathbb{Q}^\mu_\delta \cap M^\delta_\alpha$, there is $q \leqslant p$ such that $\alpha \in E^q_\delta$,
2. if $\alpha \in E^p_\delta$, then p is strongly $(\mathbb{Q}^\mu_\delta, M^\delta_\alpha)$-generic.

These two items together will imply that $\mathbb{Q}^\mu_\delta$ is strongly proper with respect to M^δ_α. The first item is again easy:

Lemma 7.2.13. *Let $\delta < \kappa^+$ and $\alpha \in \mathcal{E}_\delta$. If $p \in \mathbb{Q}^\mu_\delta \cap M^\delta_\alpha$, then there is $q \leqslant p$ with $\alpha \in E^q_\delta$.*

Proof. Exactly as Lemmas 5.2.2 and 6.2.2. $\square$

Before delving into the proof of strong properness, we record an easy version of a node density claim:

Lemma 7.2.14. *Let $\delta < \kappa^+$ and $p \in \mathbb{Q}^\mu_{\kappa^+}$. If $s \in \mathrm{dom}(f^p_\delta)$ is a node, and $p \restriction \delta$ forces ``$\bar{s} <_{\dot{S}_\delta} s$" and ``$\bar{t} <_{\dot{T}} f^p_\delta(s)$" for some nodes $\bar{s} \in \dot{S}_\delta$ and $\bar{t} \in \dot{T}$ at the same height, then $p' \leqslant p$ obtained from p by replacing f^p_δ by $f^p_\delta \cup \{(\bar{s}, \bar{t})\}$ is a condition in $\mathbb{Q}^\mu_{\kappa^+}$ that extends p.*

Proof. Clearly the resulting condition $p' \restriction \delta$ forces that $f^{p'}_\delta$ is a meet-preserving tree-embedding. It preserves the local orders too, since in case $\bar{s}$ is the immediate successor of a node s_0, and there is another node in $\bar{s}' \in S(s_0) \cap \mathrm{dom}(f^p_\delta)$, then $\bar{s}$ belongs to the effective meet of s and $\bar{s}'$, and must already be mapped to $\dot{T}$ with a local-order preserving map. Hence $S(s_0) \cap \mathsf{dom}(f^p_\delta) = \varnothing$, and the map that only maps $\bar{s} \mapsto \bar{t}$ must thus preserve the local order. $\square$

7.3 Preservation properties and node density

In this section we prove that each $\mathbb{Q}^\mu_\delta$ is κ-strongly proper, the generic embeddings $\dot{f}_\delta : S_\delta \to \dot{T}$ have the whole tree $\dot{S}_\delta$ as their domain, and that the tree $\dot{T}$ will remain in the class $\mathcal{K}_{\mu^+}$ throughout the iteration, in case none of the $\dot{S}_\delta$ has a κ-branch.

Our goal is to create an embedding from each member of $\mathcal{K}^2_{\mu^+}$ into $\dot{T}$, and indeed, we chose the bookkeeping function so that each tree $\dot{S}_\delta$ is forced to be binary, contrary to the proofs from Chapters 5 and 6. This choice plays a crucial role. It implies that whenever $s \in \dot{S}_\delta \cap M^\delta_\alpha$, then the successor set $S(s)$ is forced to be inside the model M^δ_α. Indeed, this is crucial since otherwise it might happen that there were a condition p and two nodes $s_1, s_2 \in \mathsf{dom}(f^p_\gamma) \cap S(s)$, say $s_1 \in M^\delta_\alpha$ and $s_2 \notin M^\delta_\alpha$, such that f^p_δ maps $s_1 \mapsto t_1$ and $s_2 \mapsto t_2$. With bad luck, if the tree $\dot{S}_\delta$ was not binary, it might happen that an extension w of a residue of p into M^δ_α would map some node $s^* \in S(s) \cap M^\delta_\alpha$ to some node $t^* \in T \cap M^\delta_\alpha$, such that p woud force the following about the local orders:

- $s_1 <_s s^* <_s s_2$,
- $t_1 <_{f^p_\gamma(s)} t_2 <_{f^p_\gamma(s)} t^*$.

Then w and p could not be compatible. This problem is avoided by the assumption that the trees $\dot{S}_\gamma$ are binary. This is an overkill, but since we only care about embedding partition trees into $\dot{T}$ and partition trees can always be taken binary, it does not complicate the other parts of the proof.

As said above, the required modifications in the proofs are minor and are concerned with adding two comparable nodes in the domain of an embedding approximation rather than one only. In the light of Lemma 7.2.14 it is not surprising that this can be done. Therefore, the bored reader can skip these proofs. We have anyway written them down and warmly invite the doubtful reader to read them. We have left out the proof for the preservation of Aronszajnness of $\dot{T}$, since the required modifications are essentially equal to the ones needed in node density and strong properness but the proof is considerably longer.

As said, we need different treatments for the cases when $\mu = \aleph_0$ and when μ is an uncountable successor cardinal. For the sake of simplicity, and consistency with Chapter 6, we assume from now onwards that $\mu \in \{\aleph_0, \aleph_1\}$, and divide the considerations into two subsections.

Notation 7.3.1. Denote by $\mathbb{P}_\delta^{\aleph_0}$ the poset from Definition 5.1.4 that forces the existence of a universal wide $\aleph_1$-Aronszajn tree, and denote by $\mathbb{P}_\delta^{\aleph_1}$ the poset from Definition 6.1.3 that forces the existence of a universal wide $\aleph_2$-Aronszajn tree.

Lemma 7.3.2. *Let $\delta < \kappa^+$ and $\mu \in \{\aleph_0, \aleph_1\}$. Then*

1. $\mathbb{Q}_\delta^\mu \subseteq \mathbb{P}_\delta^\mu$.
2. *If $p, q \in \mathbb{Q}_\delta$, then $q \leqslant_{\mathbb{Q}_\delta} p \iff q \leqslant_{\mathbb{P}_\delta^\mu} p$.*

Proof. Let $p \in \mathbb{Q}_\delta^\mu$. For every $\gamma < \delta$, the $\mathbb{P}_\gamma$-name $\dot{S}_\gamma$ for locally ordered binary wide κ-tree is a $\mathbb{P}_\gamma$-name for a wide κ-tree, and $p \restriction \gamma$ forces that the embedding approximation f_γ^p is an injective tree-embedding from $\dot{S}_\gamma$ into $\dot{T}$. It follows that $p \in \mathbb{P}_\delta^\mu$. If $p, q \in \mathbb{Q}_\delta^\mu$ are such that $q \leqslant_{\mathbb{Q}_\delta^\mu} p$, then again clearly $q \leqslant_{\mathbb{P}_\delta^\mu} p$. ☐

In general, the converse of Lemma 7.3.2 does not hold, and it happens that two conditions p and q are incompatible in $\mathbb{Q}_\delta^\mu$ but compatible in $\mathbb{P}_\delta^\mu$. And $\mathbb{Q}_\delta^\mu$ might fail to be a complete subposet of $\mathbb{P}_\delta^\mu$ – certainly the inclusion is not a complete embedding.

Remark 7.3.3. By Lemma 7.3.2, the following definitions make sense in the context of the posets $\mathbb{Q}_\delta^\mu$:

- trace $[p]_M$ (Definitions 5.2.5 and 6.2.4),
- the fiber notation E_γ for a set of pairs E and its corresponding successor operation (γ, β^+) (Notation 5.2.6),
- super-nice with respect to M_α^δ (Definition 6.2.9).

7.3.1 The case of $\aleph_1$

We follow closely Chapter 5.

Notation 7.3.4. In this subsection we denote $\mathbb{Q}_\delta$ for $\mathbb{Q}_\delta^{\aleph_0}$ and $\mathbb{P}_\delta$ for $\mathbb{P}_\delta^{\aleph_0}$ for every $\delta < \kappa^+$.

We begin with strong properness. We follow Subsection 5.2.2 closely. By Lemma 7.2.13 it suffices to show that if $\alpha \in E^p_\delta$, then p has a residue into M^δ_α. By Lemma 7.3.2 the following definition makes sense:

Definition 7.3.5. Let $\delta < \kappa^+$, $p \in \mathbb{Q}_\delta$ and $\alpha \in E^p_\delta$. A **residue system for p into M^δ_α (in $\mathbb{Q}_\delta$)** is a tuple $\vec{r}_E$ that is a residue system for p into M^δ_α in $\mathbb{P}_\delta$ (in the sense of Definition 5.2.7) and furthermore

$$r_{(\gamma,\beta)} \in \mathbb{Q}_\gamma \cap M^\gamma_\beta$$

for every $(\gamma, \beta) \in E$.

We have the analogue of Lemma 5.2.19:

Lemma 7.3.6. *Let $\delta < \kappa^+$, $\alpha \in \mathcal{E}_\delta$ and $p \in \mathbb{Q}_\delta$. If $\vec{r}_E$ is a residue system for p into M^δ_α in $\mathbb{Q}_\delta$, then for every $(\gamma, \beta) \in E$, the condition $r_{(\gamma,\beta)}$ is a residue for $r_{(\gamma,\beta^+)}$ into M^γ_β.*

Proof. The proof is by induction on δ. As in the proof of Lemma 5.2.19, the base case and limit case are immediate. Consider thus the successor case $\delta+1$. It suffices to show that any condition $w \in \mathbb{Q}_\delta \cap M^\delta_\alpha$ that extends the root condition $r_{(\delta,\alpha)}$ is compatible with p. We begin the construction exactly as in the proof of Lemma 5.2.19. At step $k+1$, assume that we have v_k and f_k. In addition to the requirements in the proof of Lemma 5.2.19, we assume further that for every $s' \in \mathrm{dom}(f^w_\delta)$ and $s \in X_k$, the condition v_k furthermore decides effective meet $m(s, s') = \{s \wedge s', \bar{s}', \bar{s}\}$ and the implicit image $t_{\bar{s}'}$ for the node $\bar{s}' \in m(s, s')$ that is forced to be below s'. When defining v_{k+1} and f_{k+1}, we find the condition r'_{k+1} using the Flexibility Lemma 5.2.18 (that holds in $\mathbb{Q}_\delta$ as well by the definition of a residue system in $\mathbb{Q}_\delta$). We require additionally that the there is a node $t_{\bar{s}}$ in the successor set of $t_{s\wedge s'}$ that is forced to be below $f^p_\delta(s)$ and such that the local orders are forced to satisfy

$$\bar{s}' <_{s\wedge s'} \bar{s} \iff t_{\bar{s}'} <_{t_{s\wedge s'}} t_{\bar{s}}.$$

It is possible to find such node $t_{\bar{s}}$ in $T \cap M^\delta_\alpha$ that is in $\mathrm{wd}(f^p_\delta(s)) \times \mathrm{ht}(f^p_\delta(s))$ because the size of the successor set $S(t_{s\wedge s'})$ is κ, so $|S(t_{s\wedge s'}) \cap M^\delta_\alpha| = \alpha$. Any condition decides only finite information about the order $(S(t_{s\wedge s'}), <_{t_{s\wedge s'}})$. When defining f_{k+1}, we add also the pairs $(\bar{s}', t_{\bar{s}'})$ and $(\bar{s}, t_{\bar{s}})$ for any $\bar{s}, \bar{s}'$ in the effective meet of two nodes $s \in X_k$ and $s' \in \mathrm{dom}(f^w_\delta)$. The proof is concluded in the same way as the proof of Lemma 5.2.19. ☐

Lemma 7.3.7. *Let $\delta < \kappa^+$, $\alpha \in \mathcal{E}_\delta$ and $p \in \mathbb{Q}_\delta$. If $\alpha \in E^p_\delta$, then p has a residue system into M^δ_α.*

Proof. The proof is by induction on $\delta < \kappa^+$ and exactly same as the proof of Lemma 5.2.20, with the exception that the induction hypothesis allows to choose residue systems in the poset $\mathbb{Q}_\delta$. Again the base case and the limit case are easy. At the successor case $\delta + 1$, we construct the residue system as in the proof of Lemma 5.2.20 with the stronger hypothesis that each system $\vec{r}_{E_k}$ is chosen in $\mathbb{Q}_\delta$. It follows from Lemma 7.2.14 that the condition r is a condition in $\mathbb{Q}_{\delta+1}$, and therefore the residue system $\vec{r}_E$ is a residue system for p into $M_\alpha^{\delta+1}$ in $\mathbb{Q}_\delta$. ☐

Corollary 7.3.8. *For every $\delta < \kappa^+$, the poset $\mathbb{Q}_\delta$ is strongly proper with respect to each model in the set $\{M_\alpha^\delta : \alpha \in \mathcal{E}_\delta\}$, and thus κ-strongly proper.*

Proof. Let $p \in \mathbb{Q}_\delta \cap M_\alpha^\delta$. There is $q \leqslant p$ with $\alpha \in E_\delta^q$ by Lemma 7.2.13. Let $q' \leqslant q$. Now $\alpha \in E_\delta^{q'}$. By Lemma 7.3.7 q' has a residue system $\vec{r}_E$ into M_α^δ. By Lemma 7.3.6 the root condition $r_{(\delta,\alpha)}$ is a residue for q' into M_α^δ. This shows strong properness with respect to models in $\{M_\alpha^\delta : \alpha \in \mathcal{E}_\delta\}$. The κ-strong propernes of $\mathbb{P}_\delta$ follows from the fact that we coded $\mathbb{P}_\delta$ as a subset of V_κ, using Lemma 4.1.6. ☐

We prove the node density assertion that makes sure that the embeddings $\dot{f}_\delta : \dot{S}_\delta \to \dot{T}$ will be total, i.e. satisfy $\mathsf{dom}(\dot{f}_\delta) = \dot{S}_\delta$.

Lemma 7.3.9. *Let $\delta < \kappa^+$. For any $p \in \mathbb{Q}_{\kappa^+}$ and $s \in S_\delta$ there is $q \leqslant p$ with $s \in$* $\mathsf{dom}(f_\delta^q)$.

Proof. The proof is an adaptation of the proof of Lemma 5.2.4. We may assume that $p \restriction \delta$ decides the effective meets in the set $\mathsf{dom}(f_\delta^p) \cup \{s\}$. Again, in fact, there is at most non-trivial meet that is not yet in $\mathsf{dom}(f_\delta^p)$. This is because if $s \wedge s' \neq s \wedge s''$, then these meets are directly comparable in $<_{\dot{S}}$. Say $s \wedge s'' < s \wedge s'$. But then $s \wedge s'' = (s \wedge s') \wedge s'' = s' \wedge s'' \in \mathsf{dom}(f_\delta^p)$. By Lemma 7.2.14, we may assume that this non-trivial meet, say it is $s \wedge s'$, is already in $\mathsf{dom}(f_\delta^p)$. Say the effective meet $m(s, s')$ consists of the meet $s \wedge s'$ and nodes $\bar{s}$ and $\bar{s}'$ in the successor set of $s \wedge s'$ such that

$$p \restriction \delta \Vdash ``\bar{s} \leqslant_{\dot{S}_\delta} s \text{ and } \bar{s}' \leqslant_{\dot{S}_\delta} s'".$$

Again, up to extending p to decide the implicit image of $\bar{s}'$ (the node below $f_\delta^p(s')$ at the height of $\bar{s}'$), by Lemma 7.2.14, we may assume that $\bar{s}' \in \mathsf{dom}(f_\delta^p)$.

Up to extending $p \restriction \delta$ further, we may assume that it decides the exit nodes s_α below s, where $\alpha \in N_\delta^p$, as in the proof of Lemma 5.2.4. Now the construction is exactly as in the proof of Lemma 5.2.4, with the exception that at the least step $k + 1$

with $s_{\alpha_k} \in M^{\delta}_{\alpha_{k+1}}$, when choosing the node t_{α_k}, we also choose a node $t_{\bar{s}}$ in the successor set of the meet $t_{s\wedge s'}$ in T, such that it satisfies:

- $v' \Vdash$ "$\bar{s}' <_{s\wedge s'} \bar{s} \iff f^p_\delta(\bar{s}') <_{t_{s\wedge s'}} f^p_\delta(\bar{s})$",
- if $\alpha \in \mathcal{E}_\delta$ is the least such that $t_{s\wedge s'} \in M^\delta_\alpha$, then $t_{\bar{s}} \in M^\delta_\alpha$,
- $t_{\bar{s}}$ is not forced to be below any node in $\operatorname{ran}(f^p_\delta) \cap M^\delta_{\alpha_{k+1}}$.

Such node can be found since the successor sets in the tree T are forced to be dense and have cardinality κ. We then obtain the condition v_{k+1} as in the proof of Lemma 5.2.4, with the additional assumption that v_{k+1} forces

$$t_{s\wedge s'} <_{\dot{T}} t_{\bar{s}} <_{\dot{T}} t_{\alpha_k}.$$

We finish the lemma as in the proof of Lemma 5.2.4. □

Lemma 7.3.10. *If the bookkeeping satisfies for every $\delta < \kappa^+$ that $\dot{S}_\delta$ is a $\mathbb{Q}_\delta$-name for a tree without cofinal branches, then the tree $\dot{T}$ belongs to $\mathcal{K}_\kappa$ in $V^{\mathbb{P}_{\kappa^+}}$.*

Proof. The fact that $\dot{T}$ is a wide κ-tree follows by absoluteness and the fact that its successor sets are ordered by a wide κ-Aronszajn line follows from Remark 7.2.3. The fact that it does not have a cofinal branch follows exactly as in Chapter 5. The proof is long but the needed modifications are minor, they are exactly as in the proofs of Lemmas 7.3.6 and 7.3.9, adapted in the context of the relevant proof. Therefore we skip the proof here. □

7.3.2 The case of $\aleph_2$

We closely follow Chapter 6.

Notation 7.3.11. In this subsection, we denote $\mathbb{Q}_\delta$ for $\mathbb{Q}^{\aleph_1}_\delta$ and $\mathbb{P}_\delta$ for $\mathbb{P}^{\aleph_1}_\delta$.

Here too, we begin with strong properness, move to node density, and end with preservation of Aronszajnness.

Recall that by Lemma 7.3.2, it makes sense to talk about conditions in $\mathbb{Q}_\delta$ that are super-nice with respect to a model.

Lemma 7.3.12. *Let $\delta < \kappa^+$ and $p \in \mathbb{Q}_\delta$. If $\alpha \in E^p_\delta$, then there is $q \leqslant p$ that is super-nice with respect to M^δ_α. Furthermore, the conditions that are fully super-nice are dense in $\mathbb{Q}_\delta$.*

Proof. Exactly as Lemma 6.2.12, dove-tailing with an approppriate pairing function using Lemma 7.2.14. ☐

The analogue of Lemma 6.2.13 also holds.

Lemma 7.3.13. *Let $\delta < \kappa^+$ and assume that it holds for every $\gamma < \delta$ that if $p \in \mathbb{Q}_\gamma$ is super-nice with respect to M^γ_α, then the trace $[p]_{M^\gamma_\alpha}$ is a condition in $\mathbb{Q}_\gamma \cap M^\gamma_\alpha$ and a residue of p into $\mathbb{Q}_\gamma \cap M^\gamma_\alpha$. Then it holds that if $p \in \mathbb{Q}_\delta$ is super-nice with respect to M^δ_α, then $[p]_{M^\delta_\alpha}$ is a condition in $\mathbb{Q}_\delta \cap M^\delta_\alpha$.*

Proof. Exactly as Lemma 6.2.13. ☐

In order to show strong properness, we need to prove the stronger form in terms of δ-sequences.

Definition 7.3.14. Let $\delta < \kappa^+$ and $p \in \mathbb{Q}_\delta$. A sequence $(w_j : j < \iota)$ is a ***p*-multi-extension with respect to a $\boldsymbol{\delta}$-sequence $\boldsymbol{((\delta_j, \alpha_j) : j < \iota)}$** if the following are satisfied:

1. $(w_j : j < \iota)$ is a p-multi-extension with respect to $((\delta_j, \alpha_j) : j < \iota)$ in the poset $\mathbb{P}_\delta$ (see Definition 6.2.14).
2. $w_j \in \mathbb{Q}_{\delta_j} \cap M^{\delta_j}_{\alpha_j}$ for every $j < \iota$.
3. In case $\delta_j < \delta$: for any $s \in \mathsf{dom}(f^p_{\delta_j}) \cap V_{\alpha_j}$, any
   $$s' \in \bigcup\{\mathsf{dom}(f^{w_i}_{\delta_j}) : i < j \text{ and } \delta_j < \delta_i\},$$
   and any $\beta \in \bigcup\{N^{w_i}_{\delta_j} : i < j \text{ and } \delta_j < \delta_i\}$, the condition w_j decides the effective $\dot{S}_{\delta_j}$-meet $m(s, s') = \{s \wedge s', \bar{s}, \bar{s}'\}$, where $w_j \Vdash$ ``$\bar{s} \leqslant_{\dot{S}_{\delta_j}} s$ and $\bar{s}' \leqslant_{\dot{S}_{\delta_j}} s'$", and the **implicit image of** $t_{\bar{s}'}$, i.e. the node
   $$t_{\bar{s}'} := \text{the node below } f^{w_i}_{\delta_j}(s') \text{ at the height of } \bar{s}',$$
4. if $i, j' < j$ and $s \in \mathsf{dom}(f^p_{\delta_{j'}})$, $t := f^p_{\delta_{j'}}(s) \in V_{\alpha_j}$, then for any $i < j$ such that $\delta_{j'} < \delta_i$, any
   $$(s', t') \in f^{w_i}_{\delta_{j'}},$$
   there is a node $t_{\bar{s}}$ in the successor set $S(t_{s \wedge s'})$ such that:
   (a) $w_i \Vdash_{\mathbb{Q}_{\delta_i}}$ ``$\bar{s}' <_{s \wedge s'} \bar{s}$" $\iff$ $f^{w_j}_0 \Vdash_{\mathsf{Col}(\omega_1, <\kappa)}$ ``$t_{\bar{s}'} <_{t_{s \wedge s'}} t_{\bar{s}}$",

(b) $f_0^{w_j} \Vdash_{\mathsf{Col}(\omega_1,<\kappa)}$ ``$t_{\bar{s}} <_{\dot{T}} t$".

Lemma 7.3.15. *Let $\delta < \kappa^+$ and $p \in \mathbb{Q}_\delta$. If $(w_j : j < \iota)$ is a p-multi-extension, then there is a condition $q \in \mathbb{Q}_\delta$ that extends p and each w_j, $j < \iota$.*

Proof. The proof is almost verbatim as the proof of Lemma 6.2.18. We describe the required modifications. First, note that Flexibility Lemma 6.2.16 holds for $\mathbb{Q}_\delta$ too.

The first modification is in Step 1, we add a requirement that p_1 satisfies in addition that if $i, j' < j$ and $s \in \mathsf{dom}(f^p_{\delta_{j'}})$, $t := f^p_{\delta_{j'}}(s) \in V_{\alpha_j}$, then for any $i < j$ such that $\delta_{j'} < \delta_i$, any

$$(s', t') \in f^{w_i}_{\delta_{j'}},$$

there is a node $t_{\bar{s}}$ in the successor set $S(t_{s\wedge s'})$ such that:

1. $w_i \Vdash_{\mathbb{Q}_{\delta_i}}$ ``$\bar{s}' <_{s\wedge s'} \bar{s}$" $\iff$ $f_0^{w_j} \Vdash_{\mathsf{Col}(\omega_1,<\kappa)}$ ``$t_{\bar{s}'} <_{t_{s\wedge s'}} t_{\bar{s}}$",
2. $f_0^{w_j} \Vdash_{\mathsf{Col}(\omega_1,<\kappa)}$ ``$t_{\bar{s}} <_{\dot{T}} t$".

This addition can be accommodated since the successor set $S(t_{s\wedge s'})$ is forced to be of cardinality κ and its local order is forced to be dense. Thus we may choose a node in it that w_i forces to be above $\dot{\pi}^p(t)$, and $<_{t_{s\wedge s'}}$-above $t_{\bar{s}'}$ if and only if w_i forces $\bar{s}' <_{s\wedge s'} \bar{s}$.

The next modification is in Step 2 in case $j < \iota^*$. We extend w_j even further to decide the effective meets $m(s, s') = \{s \wedge s', \bar{s}, \bar{s}'\}$, where $w_j \Vdash$ ``$\bar{s} \leqslant_{\dot{S}_{\delta_j}} s$ and $\bar{s}' \leqslant_{\dot{S}_{\delta_j}} s'$", instead of meets only. Furthermore, in addition to the implicit image of the meet $s \wedge s'$, we assume it decides the implicit image of the node $\bar{s}'$.

The third modification happens at the end of the recursion in Step 2 in case $j = \iota^*$. In the definition of w_{ι^*}, we need to add the effective meets into $f^{w_{\iota^*}}_\gamma$ for each γ, i.e. the pairs

$$(\bar{s}', t_{\bar{s}'}) \text{ and } (\bar{s}, t_{\bar{s}}),$$

where $(s, t) \in f^p_\gamma$ and $(s', t') \in \bigcup_{j<\iota^*} f^{w_j}_\gamma$.

It follows from the construction that w_{ι^*} is as wanted. □

Again, it should be noted that if p is super-nice with respect to M^δ_α and $w \in \mathbb{Q}_\delta \cap M^\delta_\alpha$ extends the trace $[p]_{M^\delta_\alpha}$, then the sequence of length one consisting of w is a p-multi-extension with respect to the δ-sequence of length one consisting of the pair (δ, α). Thus w and p have a common extension by Lemma 7.3.15

Corollary 7.3.16. *Let $\delta < \kappa^+$, $\alpha \in \mathcal{E}_\delta$ and $p \in \mathbb{Q}_\delta$.*

1. *If p is super-nice with respect to M^δ_α, then its trace $[p]_{M^\delta_\alpha}$ is a condition in $\mathbb{Q}_\delta \cap M^\delta_\alpha$ and a residue of p.*
2. *If $\alpha \in E^p_\delta$, then p is strongly $(\mathbb{Q}_\delta, M^\delta_\alpha)$-generic.*
3. *$\mathbb{Q}_\delta$ is κ-strongly proper.*

Proof. The first item follows from Lemma 7.3.15 by the observation in the above paragraph. For the second item, suppose $\alpha \in E^p_\delta$. By Lemma 7.3.12 there is $q \leqslant p$ that is super-nice with respect to M^δ_α. Then the trace $[q]_{M^\delta_\alpha}$ is a residue for q and therefore also for p into M^δ_α, by Lemma 7.3.15 and the observation in the paragraph preceding this lemma. Then we have proved that $\mathbb{Q}_\delta$ is strongly proper with respect to every model in $\{M^\delta_\alpha : \alpha \in \mathcal{E}_\delta\}$. The third item follows by Lemma 4.1.6. ☐

Next we prove node density.

Lemma 7.3.17. *Let $\delta < \kappa^+$ and $s \in \dot{S}_\delta$. For any $p \in \mathbb{Q}_{\kappa^+}$ there is $q \leqslant p$ with $s \in \mathsf{dom}(f^q_\delta)$.*

Proof. The proof follows closely the proof of Lemma 6.2.21. We again explain the required modifications. In the beginning, we extend $p \restriction \delta$ even further so that it decides for every $s' \in \mathsf{dom}(f^p_\delta)$ the effective meet $m(s, s') = \{s \wedge s', \bar{s}, \bar{s}'\}$, as well as the implicit images $t_{s \wedge s'}$ of $s \wedge s'$ and $t_{\bar{s}'}$ of $\bar{s}'$, where $\bar{s}'$ is the node in $m(s, s')$ that is forced to be below s' in the successor set of the meet $s \wedge s'$. Using Lemma 7.2.14, we may assume that $s \wedge s'$ and $\bar{s}'$ are in $\mathsf{dom}(f^p_\delta)$. Note again that if there is $s' \in \mathsf{dom}(f^p_\delta)$ that are incomparable, then there is a unique one, up to effective meet, as in the proof of Lemma 7.3.9. If there is no such s', the proof is exactly as the proof of Lemma 6.2.21. Thus we assume that there is one and we let s' be such. We denote the members of the effective meet, as decided by $p \restriction \delta$, by $s \wedge s'$, $\bar{s}'$ and $\bar{s}$. We may assume that only $\bar{s}$ is not yet added in $\mathsf{dom}(f^p_\delta)$.

Before starting the recursion, we note that the first member of the δ-sequence is a pair (δ_0, α_0) with $\delta_0 = \delta$. Instead of letting $w_0 := [p \restriction \delta_0]_{M^{\delta_0}_{\alpha_0}}$, we find $w_0 \in \mathbb{Q}_\delta \cap M^\delta_\alpha$ that extends $[p \restriction \delta]_{M^{\delta_0}_{\alpha_0}}$ and decides a node $t_{\bar{s}}$ at the height of $\bar{s}$ in the successor set of $f^p_\delta(s \wedge s')$ in T, that satisfies

$$w_0 \Vdash_{\mathbb{Q}_\delta \cap M^\delta_\alpha} ``\bar{s}' \leqslant_{s \wedge s'} \bar{s}' \iff f^p_\delta(\bar{s}') <_{f^p_\delta(s \wedge s')} t_{\bar{s}}".$$

Note that there is an exit node from M_α^δ that it below s and not in $\mathrm{dom}(f_\delta^p)$. It follows that the meet $s \wedge s'$ is in M_α^δ, and therefore its two successors $\bar{s}$ and $\bar{s}'$ also, by elementarity. Thus it is possible to find such node $t_{\bar{s}}$ using the fact that the successor set of each node in $\dot{T}$ is a dense linear order of size κ without endpoints.

Then, we proceed to define a $p \restriction \delta$-multi-extension $(w_j : j < \iota)$ exactly as in the proof of Lemma 6.2.21, with the only exception that we always find images for nodes in $\tilde{s} \in \bar{S}$ that we make to satisfy $t_{\bar{s}} <_{\dot{T}} t_{\tilde{s}}$. The proof is finished as the proof of Lemma 6.2.21, with the addition that f should also contain the pair $(\bar{s}, t_{\bar{s}})$. ☐

We have now shown κ-strong properness and node density. There remains to show that $\dot{T}$ remains Aronszajn throughout the iteration.

Lemma 7.3.18. *If the bookkeeping satisfies for every $\delta < \kappa^+$ that $\dot{S}_\delta$ is a $\mathbb{Q}_\delta$-name for a tree without cofinal branches, then the tree $\dot{T}$ belongs to $\mathcal{K}_\kappa$ in $V^{\mathbb{P}_{\kappa^+}}$.*

Proof. The fact that the successor sets of $\dot{T}$ are ordered by wide κ-Aronszajn lines follows from Remark 7.2.3. The fact that $\dot{T}$ does not have a cofinal branch follows as in Chapter 6. The proof is long but the needed modifications are minor, they are exactly as in the proofs of Lemmas 7.3.15 and 7.3.17, adapted in the context of the relevant proof. Therefore we skip the proof here. ☐

7.3.3 Conclusion

Recall the classes $\mathcal{K}_\kappa$, the class of wide κ-Aronszajn trees with wide κ-Aronszajn local orders, and $\mathcal{K}_\kappa^2$, its subclass consisting of those trees in $\mathcal{K}_\kappa$ that are binary. We have proved:

Corollary 7.3.19. *Let $\mu \in \{\aleph_1\} \cup \{\lambda^+ : \lambda \in \mathsf{Card}\}$ and assume that there is a weakly compact cardinal above μ. Then there is a forcing extension in which there is $T^* \in \mathcal{K}_{\mu^+}$ which is universal for $\mathcal{K}_{\mu^+}^2$.*

We obtain the desired conlusion:

Theorem 7.3.20. *Let $\mu \in \{\aleph_1\} \cup \{\lambda^+ : \lambda \in \mathsf{Card}\}$ and assume that there is a weakly compact cardinal above μ. Then the existence of a universal wide μ^+-Aronszajn line is consistent.*

Bibliography

[1] Omer Ben-Neria, Siiri Kivimäki, Menachem Magidor, and Jouko Väänänen. “Aronszajn trees and maximality”. In: *arXiv:2305.07880* (2025). Submitted.

[2] Keith J Devlin. “The combinatorial principle $\diamondsuit^{\#}$”. In: *The Journal of Symbolic Logic* 47.4 (1982), pp. 888–899.

[3] Mirna Džamonja and Saharon Shelah. “On properties of theories which preclude the existence of universal models”. In: *Annals of Pure and Applied Logic* 139.1-3 (2006), pp. 280–302.

[4] Mirna Džamonja and Saharon Shelah. “On wide Aronszajn trees in the presence of MA”. In: *The Journal of Symbolic Logic* 86.1 (2021), pp. 210–223.

[5] Thomas Gilton and Itay Neeman. “Side conditions and iteration theorems”. In: *Appalachian Set Theory* (2017).

[6] Tapani Hyttinen. *Games and infinitary languages.* Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes No 64, 1987.

[7] Tapani Hyttinen and Jouko Väänänen. “On Scott and Karp trees of uncountable models”. In: *The Journal of symbolic logic* 55.3 (1990), pp. 897–908.

[8] Thomas Jech. *Set theory: The third millennium edition, revised and expanded.* Springer, 2003.

[9] Akihiro Kanamori. *The higher infinite: large cardinals in set theory from their beginnings.* Springer Science & Business Media, 2008.

[10] Carol Karp. “Finite Quantifier Equivalence”. In: *Journal of Symbolic Logic.* Ed. by J. W. Addison. Amsterdam: North-Holland Pub. Co., 1965, pp. 407–412.

[11] Maaret Karttunen. *Model theory for infinitely deep languages.* Add. Acad. Sci. Fenn. Ser. A I Math. Dissertationes No. 50, 1984.

[12] Siiri Kivimäki. "Universal countably chromatic graph". In: *arxiv:2511.07608* (2025). Submitted.

[13] Siiri Kivimäki. "Universal wide Aronszajn tree". In: *arxiv:2511.06526* (2025). Submitted.

[14] Menachem Kojman and Saharon Shelah. "Nonexistence of universal orders in many cardinals". In: *The Journal of Symbolic Logic* 57.3 (1992), pp. 875–891.

[15] Georges Kurepa. "Ensembles ordonnés et leurs sous-ensembles bien ordonnés". In: *Comptes rendus hebdomadaires des séances de l'Académie des sciences* 242.18 (1956), pp. 2202–2203.

[16] Richard Laver and Saharon Shelah. "The $\aleph_2$-Souslin hypothesis". In: *Transactions of the American Mathematical Society* 264.2 (1981), pp. 411–417.

[17] Alan H Mekler. "Universal structures in power $\aleph_1$". In: *The Journal of symbolic logic* 55.2 (1990), pp. 466–477.

[18] William J Mitchell. "Adding Closed Unbounded Subsets of ω_2 with Finite Forcing". In: *Notre Dame Journal of Formal Logic* 46.3 (2005), pp. 357–371.

[19] Justin Tatch Moore. "A universal Aronszajn line". In: *Mathematical Research Letters* 16.1 (2009), pp. 121–131.

[20] Itay Neeman. "Forcing with sequences of models of two types". In: *Notre Dame Journal of Formal Logic* 55.3 (2014), pp. 265–298.

[21] Dana Scott. "Logic with denumerably long formulas and finite strings of quantifiers". In: *The theory of models*. Elsevier, 2014, pp. 329–341.

[22] Saharon Shelah. *Classification theory: and the number of non-isomorphic models.* Vol. 92. Elsevier, 1990.

[23] Saharon Shelah. *Proper and improper forcing*. Vol. 5. Cambridge University Press, 2017.

[24] Saharon Shelah. "Universal graphs without instances of CH: Revisited". English. In: *Isr. J. Math.* 70.1 (1990), pp. 69–81. ISSN: 0021-2172. DOI: `10.1007/BF02807219`.

[25] Katrin Tent and Martin Ziegler. *A course in model theory.* 40. Cambridge University Press, 2012.

[26] Stevo Todorcevic. "Lipschitz maps on trees". In: *Journal of the Institute of Mathematics of Jussieu* 6.3 (2007), pp. 527–556.

[27] Jouko Väänänen. "Games and trees in infinitary logic: A survey". In: *Quantifiers: Logics, Models and Computation: Volume One: Surveys.* Springer, 1995, pp. 105–138.